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FISCAL POLICY OPTIONS FOR REDUCING INFORMALITY IN AN ECONOMY WITH FISCAL RULES*

Tsvetomir Tsachev, Iglika Vassileva

Communicated by M. Krastanov

In memory of our friend Asen Dontchev

ABSTRACT. We use a dynamic general equilibrium model to examine the effect of various fiscal policies in an economy with a large informal sector. More specifically, we simulate (i) a reduction of the tax rate on personal income and social security contributions and (ii) an increase in the efforts of the tax administration to uncover shadow activities in a model, calibrated to the Bulgarian economy. While both policies encourage total economic activity and discourage shadow practices, they worsen the inequalities in the economy. Due to the application of fiscal rules in the model, the tax cut leads to a permanent increase in public debt and lower public investment, while the strengthened control over tax collection has a negative impact on employment, but, nonetheless, does not deepen inequalities as much as the tax cut.

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1. Introduction. This study investigates the impact of alternative policies for reduction of the size of the shadow economy. In particular, we compare the effect of an income tax cut and enhanced control and inspection measures for detection of the informal activities.

To this end we construct a dynamic general equilibrium model (DGE) with two economic sectors (formal and informal), two households (unskilled and skilled) and a government. The differentiation of skilled and unskilled households is motivated by the empirically observed concentration of informal practices in the sectors that use more intensively unskilled labour. The government can accumulate debt, but its behaviour is governed by fiscal rules - if public debt deviates from its long-term share in GDP, corrective measures are activated through adjustments in the supply of public goods and services, public investments and social transfers.

This model builds on an earlier work in [11], where a dynamic general equilibrium model with informal sector is also designed. [11], however, does not distinguish between different types of households and the specification of the government sector does not account for accumulation of public debt and for fiscal rules.

Our work is also founded on the study of Busato and Chiarini [4]. They also consider a two-sector economy with a surcharge for detected informal activities. However, their analysis is focused on the impact of the shadow economy on the business cycle, while we study the effectiveness of the policies for reduction of the size of the unobserved sector.

In [8] the impact of tax and enforcement policies on the size of the informal sector is also studied in a DSGE model with surcharge on the tax rate for uncovered shadow practices. They show that tax rate cuts are more effective in reducing the size of the informal sector than increased enforcement, but recommend combining the two policies, as increased control can partially offset the negative impact of the tax cut on the budget.

In [6] a simple DGE is constructed, intended to measure the size of the shadow economy and calibrate it to 161 countries across the world, matching various macroeconomic variables. They validate their estimates by comparing observed and model-based descriptive statistics of some key variables.

Finally, it is worth mentioning the paper [1], where the impact of budget-neutral policies on the informal sector and the economy as a whole is studied in the framework of a DSGE model. They show that neglecting the existence of the shadow sector may lead to significant distortion in the assessment of the impact

of certain policies. They also find very positive impact of the cut in corporate income taxation in the presence of a sizable informal sector.

This paper is organized as follows: the next Section 2 provides a detailed description of the model. It is followed by two technical sections, presenting the mathematical framework for the following simulations. The calibration of the model is presented in Section 5. The results of the numerical simulations are presented in Section 6. Our conclusions are summarized in Section 7 of the paper.

2. Description of the model. We consider a discrete DGE model in $T + 1$ time periods (they are numbered from 0 to T and one period corresponds to one calendar quarter), it is populated with five types of economic agents - two types of households (skilled and unskilled), two production sectors (official and informal) and a government. The skilled household provides highly qualified labour to the official production sector and holds the private capital in the economy. Meanwhile, the unskilled household supplies labour in the formal and shadow sectors. It also receives social transfers from the government.

The production sectors both operate according to a Cobb-Douglas production technology. However, the formal sector has constant returns to scale and employs skilled and unskilled labour, private and public capital, while the underground sector uses only unskilled labour and government capital to generate its output with decreasing returns to scale.

Finally, the government is characterized by its budget constraint, where public debt is accumulated when expenditures exceed revenues. It collects receipts from the unskilled and skilled officially declared labour, detected informal labour, capital and value-added taxes. Labour, profit and consumption tax rates are denoted by τ^L , τ^π and τ^C , respectively.

Government expenditures are formed from expenditures for the supply of public goods and services to the households, transfers to the unskilled household, public investment, interest on public debt and spending for uncovering underground activities. The government debt is purchased by foreign entities. For the sake of simplicity and taking into account the research questions addressed, the external sector modelling falls out of the scope of the current study.

Throughout the model we assume that there is no distinction between the product of the official and informal sectors and that it can be used both for consumption and investment. Output, consumption and investment are therefore transformed in value terms using a single price P_t , $t = 0, 1, 2, \dots, T, T + 1$.

2.1. Skilled household. The skilled household maximizes the sum of its discounted instant utilities, choosing its consumption $C_t^{P,H}$ and labour supply to the official sector H_t , taking into account its budget constraint, or, in this case, the private capital accumulation equation. The latter expresses the stock of capital in nominal terms in period $t + 1$ ($K_{t+1}P_{t+1}$) as the capital stock that has not been depreciated in period t (here δ is the rate of depreciation), plus net labour income after taxation $(1 - \tau^L)u_tH_t$ (where u_t is the wage), plus capital income r_tK_t (where r_t is the rent, i.e. the return on private capital), plus the profit of the formal sector after taxes $(1 - \tau^\pi)\pi_t$, minus a share $(1 - c_N)$ of the non-tax revenues of the government NTR_t (i.e. fees, fines, etc.), minus expenses on private consumption of the household, VAT inclusive. Part of the consumption ($\eta \in (0, 1)$) is not declared to avoid paying VAT. However some of these illegal transactions are uncovered with a probability for detection p and then a surcharge s is levied as a fine. The mathematical statement of the problem is:

$$\begin{aligned} \max_{C_t^{P,H}, H_t} \quad & \left\{ \sum_{t=0}^T \beta^t \left[\ln \left((C_t^{P,H})^{1-\zeta^H} (C_t^{G,H})^{\zeta^H} \right) + \gamma^H \ln (J^H - H_t) \right] \right\} \\ \text{s.t.} \quad & P_{t+1}K_{t+1} = P_t(1 - \delta)K_t + (1 - \tau^L)u_tH_t + r_tK_t + (1 - \tau^\pi)\pi_t \\ & - (1 - c^N)NTR_t - [(1 - \eta)(1 + \tau^C) + \eta(1 + ps\tau^C)] C_t^{P,H}P_t, \\ & 0 \leq H_t < J^H, \quad C_t^{P,H} > 0, \quad K_t \geq 0, \quad t = 0, 1, 2, \dots, T \\ & K_0 > 0 : \text{given.} \end{aligned}$$

In the above specification $\beta \in (0, 1)$ is the discount factor. The skilled household also consumes public good ($C_t^{G,H}$) and the parameter $\zeta^H \in (0, 1)$ measures the rate of substitution between private and public consumption. The parameter $\gamma^H > 0$ provides a measure of the disutility of work and J^H denotes the total available time of the skilled household, taking into account the share of the skilled population in the economy. Meanwhile, the profit π_t of the official sector is defined as

$$(1) \quad \pi_t := P_tY_t - w_tN_t - u_tH_t - r_tK_t,$$

where

$$(2) \quad Y_t := z_tK_t^{\alpha_1}(N_t^\xi H_t^{1-\xi})^{\alpha_2}G_t^{1-\alpha_1-\alpha_2}$$

is the output, N_t is the labour supply of the unskilled household to the official sector and w_t is the respective wage. In the expression for Y_t we denote by G_t

the stock of public capital at time t (see (7) below) and by z_t – the total factor productivity. The non-tax revenues of the government NTR_t are defined as a given proportion of the nominal value of the output Y_t : $NTR_t = c^{NTR} P_t Y_t$ with given coefficient $c^{NTR} \in (0, 1)$ (see (13) below). The parameters α_1 , α_2 , ξ and c_N belong to the interval $(0, 1)$.

For notational purposes we also define the aggregated consumption of the skilled household C_t^H as follows:

$$C_t^H = C_t^{P,H} + C_t^{G,H}.$$

This variable is depicted on the graph named “Consumption of skilled household” in the figure showing the simulation results.

2.2. Official production sector. The formal producer maximizes its profit, subject to its production technology:

$$(3) \quad \begin{aligned} & \max_{K_t, N_t, H_t} \{Y_t P_t - w_t N_t - u_t H_t - r_t K_t\} \\ & \text{s.t.} \quad 0 \leq H_t < J^H, \quad 0 \leq N_t < J^N, \quad K_t \geq 0, \quad K_0 > 0 : \text{given} \\ & \quad t = 0, 1, 2, \dots, T. \end{aligned}$$

Here, similar to J^H , the parameter J^N denotes the total available time of the unskilled household, based on the share of unskilled population in the economy.

2.3. Unskilled household. The unskilled household maximizes its instant utility for each of the time periods. To this end it determines its labour supply to the official and informal sectors, respectively N_t and $\tilde{N}_t$ subject to its budget constraint.

$$\begin{aligned} & \max_{N_t, \tilde{N}_t} \left\{ \ln \left((C_t^{P,N})^{1-\zeta^N} (C_t^{G,N})^{\zeta^N} \right) + \gamma^N \ln (J^N - N_t - \tilde{N}_t) \right\} \\ & \text{s.t.} \quad [(1 - \eta)(1 + \tau^C) + \eta(1 + ps\tau^C)] C_t^{P,N} P_t \\ & \quad = (1 - \tau^L) w_t N_t + (1 - ps\tau^L) \tilde{w}_t \tilde{N}_t + (1 - ps\tau^\pi) \tilde{\pi}_t + Tr_t - c^N NTR_t, \\ & \quad N_t \geq 0, \quad \tilde{N}_t \geq 0, \quad C_t^{P,N} > 0, \quad N_t + \tilde{N}_t < J^N, \\ & \quad t = 0, 1, 2, \dots, T. \end{aligned}$$

Here $\tilde{\pi}_t$ denotes the profit in the informal sector (this sector is described in the next subsection):

$$(4) \quad \begin{aligned} & \tilde{\pi}_t = \tilde{Y}_t P_t - \tilde{w}_t \tilde{N}_t \\ & \tilde{Y}_t = \tilde{z}_t \tilde{N}_t^{\psi_1} G_t^{\psi_2} \end{aligned}$$

is the output and $\tilde{z}_t$ is the total factor productivity in the informal sector. The unskilled household also consumes public good ($C_t^{G,N}$) and the parameter $\zeta^N \in (0, 1)$ measures the rate of substitution between private and public consumption. The parameter $\gamma^N > 0$ provides a measure of the disutility of work and, as mentioned in the previous subsection, J^N denotes the total available time of the unskilled household, based on the share of unskilled population in the economy. The parameters ψ_1 , and ψ_2 are strictly positive and satisfy $\psi_1 + \psi_2 < 1$.

The budget constraint of the unskilled household states that the expenses on private consumption of the household, VAT inclusive, should be equal to the labour income (wage times labour) from formally and informally supplied labour, respectively, $w_t N_t$ and $\tilde{w}_t \tilde{N}_t$ ($\tilde{w}_t$ is the wage in the informal sector), the profit $\tilde{\pi}_t$ of the informal sector plus the transfer from the government Tr_t ^{1,2}, minus a share c^N of the non-tax revenues of the government NTR_t . Similarly to the skilled household, part $\eta \in (0, 1)$ of the consumption of the unskilled household is not reported for taxation purposes. The unskilled household also tries to conceal the profit from the shadow sector. Both of these attempted tax evasion practices can be detected by the tax authorities with probability p and then a surcharge s is applied. The unskilled household, as the skilled one, consumes public good ($C_t^{G,N}$), but has no control over it; $\zeta^N \in (0, 1)$ governs the rate of substitution between private and public consumption.

Finally, similarly to the skilled household, we define the aggregated consumption of the unskilled household C_t^N as follows:

$$C_t^N = C_t^{P,N} + C_t^{G,N}$$

This variable is depicted on the graph named “Consumption of unskilled household” in the figure showing the simulation results.

2.4. Informal production sector. Similar to the official sector, the informal producer’s optimization problem can be written as³:

$$(5) \quad \begin{array}{ll} \max_{\tilde{N}_t} & \left\{ \tilde{Y}_t P_t - \tilde{w}_t \tilde{N}_t \right\} \\ \text{s.t.} & 0 \leq \tilde{N}_t < J^N \quad t = 0, 1, 2, \dots, T, \end{array}$$

¹For simplification reasons we assume that only the unskilled household receives direct transfers from the government. Given that the skilled household supplies highly qualified (and better paid) labour and owns the private capital in the economy, we can expect that the transfers from the government will be relatively small compared to its other income sources.

²Not to be mistaken for the product of T and r_t .

³In the problem of the informal sector we use the same symbols for the variables as in the official sector, but with a tilda ($\tilde{}$) sign.

where $\tilde{Y}_t$ is defined in (4) and $\tilde{w}_t$, $\tilde{N}_t$ and J^N are described in the previous subsection.

2.5. Government. The government is determined by the following eight equations, describing the dynamics of the government debt B_t ⁴, public capital G_t , public investment X_t , provision of public goods and services C_t^G , including the breakdown by the two households, and the transfers to the unskilled household Tr_t .

The government operates under fiscal rules for the public investments and goods and transfers to the households. They all adjust gradually, depending on the distance of the public debt-to-GDP ratio from its equilibrium level $\bar{B}$. The θ and the ρ parameters in equations (8)–(10) measure the extent to which the respective fiscal variables are responsive to deviations of the government debt from its long-term level. The θ -s are positive and the ρ -s are in $(0, 1)$.

$$(6) \quad B_{t+1} = (1 + \varphi)B_t + Tr_t + X_t + ap^2 + P_t C_t^G - NTR_t - \tau^L [w_t N_t + u_t H_t + ps\tilde{w}_t \tilde{N}_t] - (1 - \eta + \eta ps)\tau^C P_t [C_t^{P,N} + C_t^{P,H}] - \tau^\pi [\pi_t + ps\tilde{\pi}_t]$$

$$(7) \quad P_{t+1}G_{t+1} = (1 - \nu)P_t G_t + X_t$$

$$(8) \quad X_{t+1} = \left(\frac{X_t}{P_t Y_t} \right)^{\rho^X} \left(c^X \frac{\bar{B}}{B_t} \right)^{(1-\rho^X)\theta^X} P_t Y_t$$

$$(9) \quad Tr_{t+1} = \left(\frac{Tr_t}{P_t Y_t} \right)^{\rho^{Tr}} \left(c^{Tr} \frac{\bar{B}}{B_t} \right)^{(1-\rho^{Tr})\theta^{Tr}} P_t Y_t$$

$$(10) \quad C_{t+1}^G = \left(\frac{C_t^G}{Y_t} \right)^{\rho^{CG}} \left(c^{CG} \frac{\bar{B}}{B_t} \right)^{(1-\rho^{CG})\theta^{CG}} Y_t$$

$$(11) \quad C_t^{G,N} = c^N C_t^G$$

$$(12) \quad C_t^{G,H} = (1 - c^N)C_t^G$$

⁴With the current model specification we implicitly assume that the public debt is purchased by foreign investors without restrictions. In the case of Bulgaria, which is a small issuer of debt and has stable public finances, this assumption is considered to be realistic.

$$(13) \quad NTR_t = c^{NTR} P_t Y_t$$

The initial values B_0 , G_0 , X_0 , Tr_0 and $C_0^{G,H}$ are given and strictly positive.

The parameter φ is the interest rate on the public debt, ν is the depreciation rate of the public capital, while c^N is set at 0.69, corresponding to the proportion of unskilled labour in the Bulgarian economy⁵.

Concerning the uncovering of the shadow practices, we assume that the increase in the share of revealed informal activities is associated with a quadratic hike in the administrative costs, associated with the detection of shadow practices. In equation (6) this is represented by the term ap^2 .

3. Properties of the solutions of the four optimization problems in the model. In this section we shall show that under suitable assumptions the respective optimization problem for each of the four decision makers has an unique solution. The problems of the two production sectors have interior (for the respective feasible set) solutions. The problems of the two households have constraints for which we can not prove *a priori* that they are not binding. But taking into account that the system of the first order conditions for all the four problems is solved simultaneously, we conclude that all the constraints are not binding and thus avoid the Lagrange multipliers.

We first reformulate the problem of the skilled as well as the problem of the unskilled household. In both cases we express the consumption from the budget constraint, replace it into the objective function and take into account the properties of the $\ln(\cdot)$ function and the facts that $C_t^{G,H}$ is exogeneous for the skilled household and $C_t^{G,N}$ is exogeneous for the unskilled household. So, for the skilled household we start with

$$C_t^{P,H} = \frac{(1 - \delta)P_t K_t + (1 - \tau^L)u_t H_t + r_t K_t + (1 - \tau^\pi)\pi_t - (1 - c^N)NTR_t - P_{t+1}K_{t+1}}{P_t [(1 - \eta)(1 + \tau^C) + \eta(1 + ps\tau^C)]},$$

replace it into the objective function and as described above, we obtain the problem

$$\max_{K_t, H_t} \left\{ \sum_{t=0}^T \beta^t \left[(1 - \zeta^H) \ln \left(P_t (1 - \delta) K_t + (1 - \tau^L) u_t H_t + r_t K_t + (1 - \tau^\pi) \pi_t \right. \right. \right.$$

⁵The unskilled labour share is assumed to be the same as for the employed, where the classification of the International Labour Office has been used to identify the skilled and unskilled categories (https://www.ilo.org/ilostat-files/Documents/description_OCU_EN.pdf)

$$\begin{aligned}
& \left. - (1 - c^N) NTR_t - P_{t+1}K_{t+1} \right) + \gamma^H \ln(J^H - H_t) \Big] \Big\} \\
(14) \quad & \text{s.t. } 0 \leq H_t < J^H \text{ for } t = 0, 1, \dots, T; \ K_t \geq 0 \text{ for } t = 1, 2, \dots, T, T+1; \\
& K_0 > 0 - \text{given.}
\end{aligned}$$

In the same manner, for the unskilled household we start with

$$C_t^{P,N} = \frac{(1 - \tau^L)w_t N_t + (1 - ps\tau^L)\tilde{w}_t \tilde{N}_t + (1 - ps\tau^\pi)\tilde{\pi}_t + Tr_t - c^N NTR_t}{P_t [(1 - \eta)(1 + \tau^C) + \eta(1 + ps\tau^C)]}$$

and obtain the problem

$$\begin{aligned}
& \max_{N_t, \tilde{N}_t} \quad \left\{ (1 - \zeta^N) \ln \left((1 - \tau^L)w_t N_t + (1 - ps\tau^L)\tilde{w}_t \tilde{N}_t + \right. \right. \\
& \quad \left. \left. (1 - ps\tau^\pi)\tilde{\pi}_t + Tr_t - c^N NTR_t \right) + \gamma^N \ln(J^N - N_t - \tilde{N}_t) \right\} \\
(15) \quad & \text{s.t.} \\
& N_t \geq 0, \ \tilde{N}_t \geq 0, \ N_t + \tilde{N}_t < J^N, \text{ for } t = 0, 1, 2, \dots, T.
\end{aligned}$$

For each time period t , $t = 0, 1, \dots, T$, we have to determine eight variables (H_t , K_{t+1} , N_t , $\tilde{N}_t$, u_t , r_t , w_t and $\tilde{w}_t$) from the eight necessary optimality conditions (two conditions for the skilled household's problem, two for the unskilled household's, three for the official sector's and one for the informal sector's problem). Because of their economic meaning H_t , K_t , N_t and $\tilde{N}_t$ are constrained in the two households' problems to be non-negative. The other four variables, u_t , r_t , w_t and $\tilde{w}_t$, having the meaning of prices, have to be non-negative too. But there is an obvious mathematical argument that in equilibrium (if it exists) the last four ones are strictly positive. Indeed, the official production sector's problem has no solution if $u_t \leq 0$, or $r_t \leq 0$, or $w_t \leq 0$ because its objective function becomes strictly increasing either in H_t , or in K_t , or in N_t respectively. The same argument applies to $\tilde{w}_t$ and $\tilde{N}_t$ because of the informal sector's problem. Formulating the claims in the last two sentences we tacitly used the fact that the upper bounds imposed on labour are strict ($< J^H$ and $< J^N$ respectively) and that K_t is not bounded from above. Also, we shall ignore the (strict) upper bounds on labour in the optimization problems (this is equivalent to assuming that they are large enough). After the equilibrium is found numerically with the values of the parameters as we have calibrated them, it will turn out that these constraints are satisfied.

3.1. The two production sectors. The value function of the official producer's problem (i.e. its profit) is

$$\pi_t(N_t, H_t, K_t) = P_t z_t K_t^{\alpha_1} N_t^{\alpha_2 \xi} H_t^{\alpha_2(1-\xi)} G_t^{1-\alpha_1-\alpha_2} - w_t N_t - u_t H_t - r_t K_t$$

and is strictly concave (Claim 4 in Appendix B) with respect to (N_t, H_t, K_t) on $\{(N_t, H_t, K_t) \in \mathbf{R}^3 : N_t \geq 0, H_t \geq 0, K_t \geq 0\}$. As mentioned in the introduction of this section, when looking for an equilibrium, we have to consider u_t , r_t and w_t to be positive. Then, applying again Claim 4 in Appendix B, we conclude that $\pi_t(\cdot, \cdot, \cdot)$ has unique global maximum on $\{(H_t, K_t, N_t) \in \mathbf{R}^3 : H_t \geq 0, K_t \geq 0, N_t \geq 0\}$. Next, since

$$\lim_{N_t \downarrow 0} \frac{\partial \pi_t}{\partial N_t} = \lim_{H_t \downarrow 0} \frac{\partial \pi_t}{\partial H_t} = \lim_{K_t \downarrow 0} \frac{\partial \pi_t}{\partial K_t} = +\infty,$$

this global maximum is attained in $\{(H_t, K_t, N_t) \in \mathbf{R}^3 : H_t > 0, K_t > 0, N_t > 0\}$. So, we have

Proposition 1. *Assume that J^N and J^H are large enough. Then Problem (3) has unique solution which is attained at an interior point of the feasible set.*

From here it follows immediately that in equilibrium (if it exists) we must have $H_t > 0$, $N_t > 0$ and $K_t > 0$. Also, the first order necessary and sufficient optimality conditions for the official producer's problem are

$$\begin{aligned} w_t &= P_t \alpha_2 \xi z_t K_t^{\alpha_1} N_t^{\alpha_2 \xi - 1} H_t^{\alpha_2(1-\xi)} G_t^{1-\alpha_1-\alpha_2} \\ u_t &= P_t \alpha_2 (1 - \xi) z_t K_t^{\alpha_1} N_t^{\alpha_2 \xi} H_t^{\alpha_2(1-\xi) - 1} G_t^{1-\alpha_1-\alpha_2} \\ r_t &= P_t \alpha_1 z_t K_t^{\alpha_1 - 1} N_t^{\alpha_2 \xi} H_t^{\alpha_2(1-\xi)} G_t^{1-\alpha_1-\alpha_2} \end{aligned}$$

which implies

$$\begin{aligned} w_t N_t &= P_t \alpha_2 \xi z_t K_t^{\alpha_1} N_t^{\alpha_2 \xi} H_t^{\alpha_2(1-\xi)} G_t^{1-\alpha_1-\alpha_2} \\ u_t H_t &= P_t \alpha_2 (1 - \xi) z_t K_t^{\alpha_1} N_t^{\alpha_2 \xi} H_t^{\alpha_2(1-\xi)} G_t^{1-\alpha_1-\alpha_2} \\ r_t K_t &= P_t \alpha_1 z_t K_t^{\alpha_1} N_t^{\alpha_2 \xi} H_t^{\alpha_2(1-\xi)} G_t^{1-\alpha_1-\alpha_2} \end{aligned} \tag{16}$$

and, because of (1),

$$\pi_t = (1 - \alpha_1 - \alpha_2) P_t z_t K_t^{\alpha_1} N_t^{\alpha_2 \xi} H_t^{\alpha_2(1-\xi)} G_t^{1-\alpha_1-\alpha_2} \tag{17}$$

Remark 1. The fact that in equilibrium (if it exists) we have $H_t > 0$, $N_t > 0$ and $K_t > 0$ and equations (7)–(10) imply that iteratively we obtain $G_t > 0$, $X_t > 0$, $Tr_t > 0$ and $C_t^G > 0$ for $t = 0, 1, 2, \dots, T$.

Applying exactly the same argument as above, we have

Proposition 2. *Assume that J^N is large enough. Then Problem (5) has unique solution which is attained at an interior point of the feasible set.*

So, in equilibrium (if it exists) we must have $\tilde{N}_t > 0$. Also, the first order necessary and sufficient optimality condition for the official producer's problem is

$$\tilde{w}_t = P_t \psi_1 \tilde{z}_t \tilde{N}_t^{\psi_1 - 1} G_t^{\psi_2}$$

which implies

$$(18) \quad \tilde{w}_t \tilde{N}_t = P_t \psi_1 \tilde{z}_t \tilde{N}_t^{\psi_1} G_t^{\psi_2}.$$

3.2. The skilled household. To shorten the notations, we set $M_t := (1 - \tau^\pi)\pi_t - (1 - c^N)NTR_t$. This exogeneous for the skilled household variable is positive in equilibrium because it equals $[(1 - \tau^\pi)(1 - \alpha_1 - \alpha_2) - (1 - c^N)c^{NTR}]P_t Y_t$ (we have used the notation (2)) and the values of the data (see Section “Calibration”) ensure that $M_t > 0$. Then we rewrite the skilled household's problem (14) as

$$(19) \quad \begin{aligned} \max_{K_t, H_t} & \left\{ \sum_{t=0}^T \beta^t \left[(1 - \zeta^H) \ln \left(P_t(1 - \delta)K_t + (1 - \tau^L)u_t H_t + r_t K_t + M_t - P_{t+1}K_{t+1} \right) \right. \right. \\ & \left. \left. + \gamma^H \ln(J^H - H_t) \right] \right\} \\ \text{s.t.} & \quad 0 \leq H_t < J^H \text{ for } t = 0, 1, \dots, T; K_t \geq 0 \text{ for } t = 1, 2, \dots, T, T + 1; \\ & \quad K_0 > 0 - \text{given.} \end{aligned}$$

Proposition 3. *Problem (19) has unique solution.*

A sketch of the proof is given in Appendix B.

3.3. The unskilled household. As in the case with the skilled household, introducing the exogeneous for the unskilled household variable $Q_t := (1 - ps\tau^\pi)\tilde{\pi}_t + Tr_t - c^N NTR_t$, we rewrite the unskilled household's problem as

$$\begin{aligned}
& \max_{N_t, \tilde{N}_t} \left\{ (1 - \zeta^N) \ln \left((1 - \tau^L) w_t N_t + (1 - p s \tau^L) \tilde{w}_t \tilde{N}_t + Q_t \right) \right. \\
& \qquad \qquad \qquad \left. + \gamma^N \ln(J^N - N_t - \tilde{N}_t) \right\} \\
(20) \quad & \text{s.t. } N_t \geq 0, \quad \tilde{N}_t \geq 0, \quad N_t + \tilde{N}_t < J^N, \quad \text{for } t = 0, 1, 2, \dots, T.
\end{aligned}$$

Proposition 4. *If the feasible set of Problem (20) is nonempty, then the Problem has unique solution.*

A sketch of the proof is given in Appendix C.

Remark 2. The values of the data (see Section “Calibration”) we use ensure that the feasible set of Problem (20) is nonempty for $t = 0, 1, \dots, T$. This is due to the fact that when solving the simultaneous system of first order conditions from (16) we obtain

$$(1 - \tau^L) w_t N_t - c^N N T R_t = [(1 - \tau^L) \alpha_2 \xi - c^N c^{N T R}] P_t Y_t > 0.$$

4. Necessary and sufficient conditions for the two households’ problems.

Remark 3. As mentioned at the beginning of Section (3), the problems of the two households have constraints for which we can not prove *a priori* that they are not binding:

$$0 \leq H_t, \quad 0 \leq K_t, \quad 0 \leq N_t, \quad \text{and } 0 \leq \tilde{N}_t.$$

However, since the system of the first order conditions for all the four problems is solved simultaneously, all the constraints can be considered to be not binding and thus we can avoid the Lagrange multipliers.

4.1. Necessary and sufficient conditions for the skilled household problem. Let us denote the objective function of Problem (19) by Φ . Differentiating it we obtain the following necessary conditions, which are also sufficient, since Proposition 3 claims that the attained solution is unique.

$$\begin{aligned}
(21) \quad & \frac{1}{\beta^t} \frac{\partial \Phi}{\partial H_t} \\
& = \frac{(1 - \zeta^H)(1 - \tau^L) u_t}{P_t(1 - \delta) K_t + (1 - \tau^L) u_t H_t + r_t K_t + (1 - \tau^\pi) \pi_t - (1 - c^N) N T R_t - P_{t+1} K_{t+1}} \\
& \qquad \qquad \qquad - \frac{\gamma^H}{J^H - H_t} = 0, \quad t = 0, 1, \dots, T.
\end{aligned}$$

$$\begin{aligned}
(22) \quad & \frac{1}{\beta^{t-1}} \frac{1}{1 - \zeta^H} \frac{\partial \Phi}{\partial K_t} \\
&= \frac{\beta[(1 - \delta)P_t + r_t]}{(1 - \delta)P_t K_t + (1 - \tau^L)u_t H_t + r_t K_t + (1 - \tau^\pi)\pi_t - (1 - c^N)NTR_t - P_{t+1}K_{t+1}} \\
&\quad - \frac{P_t}{(1 - \delta)P_{t-1}K_{t-1} + (1 - \tau^L)u_{t-1}H_{t-1} + r_{t-1}K_{t-1} + (1 - \tau^\pi)\pi_{t-1} - (1 - c^N)NTR_{t-1} - P_t K_t} \\
&= 0, \quad t = 1, 2, \dots, T.
\end{aligned}$$

4.2. Necessary and sufficient conditions for the unskilled household problem. Let us denote the objective function of Problem (20) by Ψ . Similarly to the skilled household, the necessary and sufficient conditions for the unskilled household are obtained by differentiation of the objective function of Problem (20), taking into account Proposition 4.

$$\begin{aligned}
(23) \quad & \frac{\partial \Psi}{\partial N_t} = \frac{(1 - \zeta^N)(1 - \tau^L)w_t}{(1 - \tau^L)w_t N_t + (1 - ps\tau^L)\tilde{w}_t \tilde{N}_t + (1 - ps\tau^\pi)\tilde{\pi}_t + Tr_t - c^N NTR_t} \\
&\quad - \frac{\gamma^N}{J^N - N_t - \tilde{N}_t} = 0 \quad \text{for } t = 0, 1, \dots, T,
\end{aligned}$$

$$\begin{aligned}
(24) \quad & \frac{\partial \Psi}{\partial \tilde{N}_t} = \frac{(1 - \zeta^N)(1 - ps\tau^L)\tilde{w}_t}{(1 - \tau^L)w_t N_t + (1 - ps\tau^L)\tilde{w}_t \tilde{N}_t + (1 - ps\tau^\pi)\tilde{\pi}_t + Tr_t - c^N NTR_t} \\
&\quad - \frac{\gamma^N}{J^N - N_t - \tilde{N}_t} = 0 \quad \text{for } t = 0, 1, \dots, T.
\end{aligned}$$

Equations (21)–(24), together with the first order conditions of the formal and informal sectors, equations (16) and (18), respectively, and the government policy and constraint equations (6)–(10) constitute a nonlinear system of $13 * (T + 1) - 6$ equations with $13 * (T + 1) - 6$ unknowns⁶, which give the solution to the model.

⁶The unknown variables include

- (i) $H_t, N_t, \tilde{N}_t, u_t, w_t, \tilde{w}_t, r_t$ for $t = 0, \dots, T$ and
- (ii) $K_t, B_t, G_t, X_t, Tr_t, C_t^G$ for $t = 1, \dots, T$. The initial values of these variables are given (and are strictly positive), and, as shown in Appendix B, at the maximum $K_{T+1} = 0$.

5. Calibration. The model parameters are calibrated as described in Table 1. The parameters in the official sector production function α_1 , α_2 and ξ are set at respectively 0.45, 0.35 and 0.55, where the technology is with constant returns to scale. Thus, we indirectly arrive at an elasticity of official output to public capital of 0.20, which slightly higher but close to what is typically estimated in economic literature⁷.

Concerning the informal sector, its parameters ψ_1 and ψ_2 are set at 0.90 and 0.09, respectively, implying decreasing returns to scale and much lower elasticity to public capital. Decreasing returns to scale, in particular, imply that if expanded, the production process in the informal sector becomes less efficient. This is a logical assumption, given that larger-scale shadow sector production is typically associated with higher costs to avoid detection.

Table 1. Parameter values

Parameter	Value	Parameter	Value
β	0.95	p	0.13
δ	0.06	s	1.50
ν	0.04	<i>nrabudget</i>	0.00191916
α_1	0.45	τ^C	0.20
α_2	0.35	τ^L	0.29
ξ	0.55	τ^π	0.16
ψ_1	0.90	ρ^X	0.95
ψ_2	0.09	ρ^{Tr}	0.95
<i>vatgap</i>	8.3%	ρ^{CG}	0.95
$\bar{B}$	0.20	θ^X	0.10
ζ^N	0.21	θ^{Tr}	0.10
ζ^H	0.21	θ^{CG}	0.10
J^N	0.2670	c^N	0.69
J^H	0.1199	c^{NTR}	0.052

The tax rates are defined so as to achieve statistically observed average share of the respective revenues in GDP. Consumption taxes include VAT, excises and import duties, taxes on labour include the personal income tax and the social security contribution, while the capital tax covers profit and dividend taxes and other taxes. The variables governing the responsiveness of each of the expenditure items (ρ^X , ρ^{Tr} , ρ^{CG} and θ^X , θ^{Tr} , θ^{CG}) are calibrated at values that produce

⁷See, for example [5]

reasonable variability of the expenditures in time and depending on the deviation of the public debt from its long-term level.

The parameters, governing the probability of detection of shadow activities p and applied surcharge in case of discovery s are set at 13% and 1.5, respectively. The probability of detection is based on estimates of the National Revenue Agency about the total loss for the budget as a result of informal activities and the part of this loss which is detected (see [3]). The surcharge s is determined based on the ratio of the fine if detected for the first time to the average wage in Bulgaria. Finally, the expenditure parameter a , associated with the efforts to uncover the shadow economy, is determined to reproduce the tax administration expenditures-to-GDP ratio $nrabudget$.

The share of the informal consumption transactions η is determined based on an EC estimate of the VAT gap $vatgap$ (see [7]) and the resulting estimate of η amounts to 10.3%.

The parameters in the household problems are calibrated as follows: the shares of public consumption for both households ζ^N and ζ^H are calibrated at 0.21, which is the share of public to total consumption, while the disutility-of-work parameters γ^N and γ^H are calibrated to make sure that the respective first order conditions for the official unskilled and skilled labor supply hold in the steady state. The same approach is applied also with respect to the total factor productivity measures z_t and $\tilde{z}_t$, which are determined so that the respective production functions definitions are met at the steady state. Finally, the derivation of the values of the total time endowment of the two types of households and the steady state values of the skilled and unskilled official labour are given in Appendix A.

6. Numerical simulations. The dynamic general equilibrium model, described in the previous three sections is implemented in **Python**⁸. Python is an open-source interpreted, high-level, general-purpose programming language that is generally perceived to achieve good balance between ease of use and high performance.

Similar to other open-source languages, such as R and Julia, Python has a package manager that allows the users to develop and share packages, intended to solve specific problems. In this paper we use the **Dolo** package⁹. It provides an easy-to-apply tool for specification and solution of rational expectations economic models.

⁸See [10], downloadable at <https://www.python.org/>.

⁹<https://github.com/EconForge/dolo.py>

In Dolo, models are written in simple text files, using YAML (Yet Another Markup Language). This language allows the user to specify complex structures in a human readable way. The YAML files are run in Jupiter notebooks, which are typically used to create and share documents, containing simultaneously code, text, formulas and visualizations.

Dolo also features sophisticated numerical routines, intended to solve the specified models, such as perfect forecast, policy iteration and local perturbation methods. We apply the perfect foresight method. It boils down to solving a non-linear stacked system of equations for all time periods in the simulation horizon. It requires the specification of an initial state. In our case we assume that the model is initially in the steady state, calculated analytically in Appendix D.

6.1. Alternative policy simulations. The figures below illustrate the dynamics of the main macroeconomic variables and the new steady state in the model in two cases:

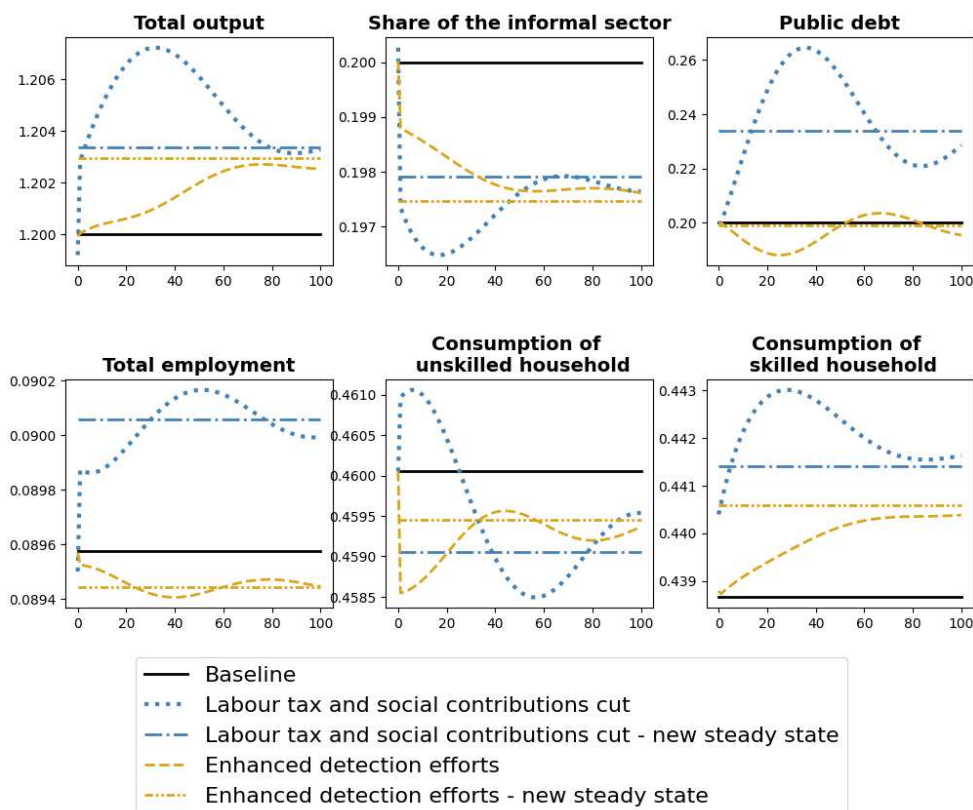
1. Decrease in the tax rate on income and social security contributions τ^L by 1 percentage point and
2. Increase in the efforts for uncovering the informal activities, which are reflected in a hike in the probability of detection of the shadow sector by 1 percentage point.

The impact of both policies on total economic activity is positive and, at the same time, leads to a decline in the gray economy. However the transmission mechanisms between the two simulations vary significantly.

The **reduction in the rate of the personal income tax and/ or social security contributions** worsens the fiscal stance (i.e. increases the public debt), but stimulates labour supply. As the effect on the official sector is larger, there is transfer of labour force from the informal to the official sector. Lower taxes increase the net remuneration for both types of labour and, despite the decline in informal work, total employment in the economy is higher.

As the public finances worsen, the application of the fiscal rules leads to a decline in public capital, transfers and the provision of the public good. As a result, despite higher labour income, the consumption of the unskilled household declines, as it relies also on social transfers. In contrast, the consumption of the skilled household is boosted due to higher labour and capital incomes.

Stepping up the fiscal control over the informal activities supports revenue growth and, despite higher expenditures for enhancing tax collection, leads to a marginal improvement in the public balance and debt. This in turn activates the fiscal rules in the opposite direction leaving room for higher public investment, provision of public goods and transfers. As a result, the decline in



the consumption of the unskilled household in this case is lower compared to the tax cut simulation.

Another major difference with the tax rate reduction is that higher fiscal control suppresses labour demand in the shadow sector, which brings the wages of the unskilled labour down. Overall total employment declines due to higher reduction in the unskilled work in the informal sector as compared to the increase in the official sector and no impact on the skilled labour.

To sum up both policies result in higher consumption, private investment and output in the economy and reduce the size of the informal economy. However, they have opposite effects on the public finances and therefore, following the fiscal rules, on the public capital accumulation, social transfers and provision of public goods.

Additionally, while tax reduction stimulates labour supply, stricter fiscal control suppresses labour demand. As a result, total employment in the economy

Table 2. Estimated effects in policy simulations (% changes, compared to the baseline)

Variable	Labour tax rate cut	Enhanced fiscal control
Total output	0.3%	0.2%
Official output	0.5%	0.5%
Total investment	0.2%	0.5%
Private investment	0.5%	0.5%
Public investment	-1.1%	0.5%
Total consumption	0.2%	0.1%
Consumption of unskilled household	-0.2%	-0.1%
Consumption of skilled household	0.6%	0.4%
Total employment	0.5%	-0.1%
Labour productivity	-0.3%	0.4%
Public debt	17.0%	-0.9%

increases in the former case and declines in the latter. Since the impact on the output is similar, the effect on labour productivity is negative in the tax cut simulation and positive when stricter fiscal control is applied.

Finally, both policies have similar, positive impact on total consumption, but lead to deepening inequalities as the decline in the consumption of the unskilled (usually poorer) household is compensated by an increase in the consumption of the skilled (usually richer) household. This problem is more exacerbated when a tax cut is applied.

7. Conclusion. The presented dynamic general equilibrium model with fiscal rules is built specifically for the purpose of studying the impact of fiscal policies on the informal sector and is calibrated to the Bulgarian economy. Both simulated policies, an income tax cut and strengthened control over tax collection, encourage the formal economic activity at the expense of the undeclared one, but lowering the tax rate leads to an overall increase in employment, while improving the tax collection is associated with a decline. Meanwhile, a tax cut lowers revenues, while enhanced control boosts tax collection rate, thus the first worsens the public finances, while the latter improves them. As a result, with the higher control, public investment increases, while the tax cut has a negative impact on public capital spending. Finally, both policies lead to deepening income inequalities in the economy, but they are more pronounced when an income tax rate cut is implemented.

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Appendix

A. Derivation of the values of the households' time endowments and skilled and unskilled official labour at the steady state.

Bulgarian GDP amounted to BGN 120.396 bn in 2019, which is equivalent to BGN 30.09875 bn on average per quarter. This output was created with an input of 1 453 402 575 hours worked. Equivalently, we can say that BGN 1 GDP used $\frac{1.453402575}{30.09875} = 0.0482$ hours worked. Taking into account that the share of the unskilled and skilled labour are, respectively, 69% and 31%, then we can calibrate the total hours of the unskilled and skilled households at $\mathbf{N} = \mathbf{0.0333}$ and $\mathbf{H} = \mathbf{0.0149}$.

Separately, total available time to each individual is 8 760 hours per year (24 hours per day x 365 days per year), which is equivalent to 2190 hours per quarter¹⁰. Taking into account that the size of the working age population in the country is 5 327 043 people¹¹, then the total available time in the economy per quarter is 11 666 224 170 hours.

Meanwhile, the Bulgarian National Statistical Institute reported 5 813 610 thousand hours worked per annum in 2019, i.e. 1 453 402 575 hours worked per quarter. Therefore, the share of labour in total available time in the Bulgarian economy is 12.458%.

Therefore, since the total hours worked per one unit of GDP are estimated at 0.0482, the total available time, rescaled to a unit of GDP will be $\frac{0.0482}{0.12458} = 0.3869$. Again using the same distribution between the unskilled and skilled households, we obtain $\mathbf{J}^{\mathbf{N}} = \mathbf{0.2670}$ and $\mathbf{J}^{\mathbf{H}} = \mathbf{0.1199}$.

B. Properties of the skilled household's problem. We shall use the following obvious claims:

Claim 1. Let S be a convex subset of $\mathbf{R}^m$, $\Delta \subset \mathbf{R}^1$ be an interval, $g(\cdot) : S \rightarrow \Delta$ be concave and $f(\cdot) : \Delta \rightarrow \mathbf{R}^1$ be increasing and strictly concave. Then $s(x) := f(g(x))$ is strictly concave on S .

Claim 2. Let S be a convex subset of $\mathbf{R}^m$ and $f(\cdot) : S \rightarrow \mathbf{R}^1$ and $g(\cdot) : S \rightarrow \mathbf{R}^1$ be concave functions, one of which is strictly concave. Then $h(x) := f(x) + g(x)$ is strictly concave on S .

¹⁰The model is calibrated to quarterly data for the Bulgarian economy.

¹¹Data from population statistics of the National Statistical Institute for 2019.

Claim 3. Let S_1 be a convex set in $\mathbf{R}^m$, S_2 be a convex set in $\mathbf{R}^n$ and S_3 be a convex set in $\mathbf{R}^p$. Let $f(\cdot, \cdot)$ be a strictly concave function on $S_1 \times S_2$ and $g(\cdot, \cdot)$ be a strictly concave function on $S_2 \times S_3$. Then $\phi(x, y, z) := f(x, y) + g(y, z)$ is strictly concave on $S_1 \times S_2 \times S_3$.

Claim 4. If $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha + \beta + \gamma < 1$, $A > 0$ and a , b and c are arbitrary constants, the function $f(x, y, z) = Ax^\alpha y^\beta z^\gamma - ax - by - cz$ is strictly concave on $\{(x, y, z) \in \mathbf{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$. If, in addition, $a > 0$, $b > 0$ and $c > 0$, $f(\cdot, \cdot, \cdot)$ has unique global maximum on $\{(x, y, z) \in \mathbf{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$.

Proof of Proposition 3.

Step 1: The feasible set of Problem (19) is nonempty. Next to the constraints $0 \leq H_t < J^H$ for $t = 0, 1, \dots, T$ and $K_t \geq 0$ for $t = 1, 2, \dots, T, T+1$, any feasible point

$(\tilde{H}_0, \tilde{H}_1, \dots, \tilde{H}_T; \tilde{K}_1, \tilde{K}_2, \dots, \tilde{K}_T, \tilde{K}_{T+1})^{12}$ for Problem (19) has to satisfy also

$$(1 - \delta)P_t \tilde{K}_t + (1 - \tau^L)u_t \tilde{H}_t + r_t \tilde{K}_t + M_t - P_{t+1} \tilde{K}_{t+1} > 0$$

for $t = 0, 1, \dots, T$. Now, for $t = 0, 1, \dots, T$ take any $\tilde{H}_t$ in $[0, J^H)$ and then consecutively for $t = 1, 2, \dots, T, T+1$ take any $\tilde{K}_t > 0$ which satisfies

$$\tilde{K}_t < \frac{1}{P_t} \left((1 - \delta)P_{t-1} \tilde{K}_{t-1} + (1 - \tau^L)u_{t-1} \tilde{H}_{t-1} + r_{t-1} \tilde{K}_{t-1} + M_{t-1} \right)$$

(here we have in mind that $\tilde{K}_0 := K_0 > 0$ is given). Obviously,

$$(\tilde{H}_0, \tilde{H}_1, \dots, \tilde{H}_T; \tilde{K}_1, \tilde{K}_2, \dots, \tilde{K}_T, \tilde{K}_{T+1})$$

is feasible for Problem (19).

Step 2: At the point of maximum we have $K_{T+1} = 0$. This is due to the fact that the cost function of Problem (19) is strictly decreasing with respect to K_{T+1} .

Step 3: Problem (19) has at least one solution. To prove this we shall show that outside of a compact subset of the feasible set of Problem (19) its cost function Φ takes values which are low enough.

Starting with $\bar{K}_0 := K_0$ we consecutively define the numbers

$$\bar{K}_t = \frac{1}{P_t} \left((1 - \delta)P_{t-1} \bar{K}_{t-1} + (1 - \tau^L)u_{t-1} J^H + r_{t-1} \bar{K}_{t-1} + M_{t-1} \right)$$

¹²The notation “ $\cdot$ ” is used in Appendix B for denoting a specified feasible point for Problem (19) and is not related to the unskilled household's problem.

for $t = 1, 2, \dots, T, T+1$. So, if $(H_0, H_1, \dots, H_T; K_1, K_2, \dots, K_T, K_{T+1})$ is a feasible point, we have $H_t \in [0, J^H]$ for $t = 0, 1, \dots, T$ and $K_t \in (0, \bar{K}_t)$ for $t = 1, 2, \dots, T, T+1$.

Next, choose $\nu > 0$ such that

$$\begin{aligned} \sum_{i=0, i \neq t}^T & \left[\beta^i (1 - \zeta^H) \ln \left((1 - \delta) P_i \bar{K}_i + (1 - \tau^L) u_i J^H + r_i \bar{K}_i + M_i \right) + \gamma^H \ln J^H \right] \\ & + \beta^t (1 - \zeta^H) \ln \left((1 - \delta) P_t \bar{K}_t + (1 - \tau^L) u_t J^H + r_t \bar{K}_t + M_t \right) \\ & + \gamma^H \ln (J^H - H_t) < \Phi(\tilde{H}_0, \tilde{H}_1, \dots, \tilde{H}_T; \tilde{K}_1, \tilde{K}_2, \dots, \tilde{K}_T, \tilde{K}_{T+1}) \end{aligned}$$

for $t \in \{0, 1, \dots, T\}$ and $J^H - \nu < H_t < J^H$. Then, for each feasible $(H_0, H_1, \dots, H_T; K_1, K_2, \dots, K_T, K_{T+1})$ with $J^H - \nu < H_t < J^H$ for some $t \in \{0, 1, \dots, T\}$ we have

$$\begin{aligned} \Phi(H_0, H_1, \dots, H_T; K_1, K_2, \dots, K_T, K_{T+1}) \\ < \Phi(\tilde{H}_0, \tilde{H}_1, \dots, \tilde{H}_T; \tilde{K}_1, \tilde{K}_2, \dots, \tilde{K}_T, \tilde{K}_{T+1}), \end{aligned}$$

i.e. the maximum of Φ can not be attained if $J^H - \nu < H_t < J^H$ for some $t \in \{1, 2, \dots, T\}$.

We next show that for each $t \in \{1, 2, \dots, T\}$ there exists a continuous function $\epsilon_t(\cdot) : [0, J^H - \nu] \rightarrow (0, \bar{K}_t)$ such that the maximum of Φ can not be attained at values of (H_{t-1}, K_t) for which $K_t > \epsilon_t(H_{t-1})$ holds. To this end we consider the terms of Φ containing K_t and replace in them K_{t-1} by $\bar{K}_{t-1}$ and H_t by J_H , namely

$$\begin{aligned} (25) \quad & \beta^{t-1} (1 - \zeta^H) \ln \left((1 - \delta) P_{t-1} \bar{K}_{t-1} + (1 - \tau^L) u_{t-1} H_{t-1} + r_{t-1} \bar{K}_{t-1} + M_{t-1} - P_t K_t \right) \\ & + \beta^t (1 - \zeta^H) \ln \left((1 - \delta) P_t K_t + (1 - \tau^L) u_t J^H + r_t K_t + M_t - P_{t+1} K_{t+1} \right). \end{aligned}$$

Skipping $-P_{t+1} K_{t+1}$ from the second term above and introducing in an obvious manner the strictly positive constants A_t , B_t , C_t and D_t (in order to shorten the notations), we see that (25) does not exceed

$$\varphi(H_{t-1}, K_t) := \beta^{t-1} (1 - \zeta^H) \ln \left(A_t + B_t H_{t-1} - P_t K_t \right) + \beta^t (1 - \zeta^H) \ln \left(C_t + D_t K_t \right)$$

which is strictly concave on $[0, J^H - \nu] \times \left[0, \frac{1}{P_t} (A_t + B_t H_{t-1}) \right)$ because of Claims 1 and 2.

Now we note that for each $H_{t-1} \in [0, J^H - \nu]$ we have

$$\lim_{K_t \uparrow \frac{1}{P_t}(A_t + B_t H_{t-1})} \frac{\partial \varphi}{\partial K_t}(H_{t-1}, K_t) = -\infty, \quad \lim_{K_t \uparrow \frac{1}{P_t}(A_t + B_t H_{t-1})} \varphi(H_{t-1}, K_t) = -\infty$$

and that $\frac{\partial \varphi}{\partial K_t}(H_{t-1}, \cdot)$ is strictly decreasing on $\left(0, \frac{1}{P_t}(A_t + B_t H_{t-1})\right)$. Applying the Implicit function theorem we find a continuous function $\phi(\cdot) : [0, J^H - \nu] \rightarrow \left(0, \frac{1}{P_t}(A_t + B_t H_{t-1})\right)$ such that $\frac{\partial \varphi}{\partial K_t}(H_{t-1}, \phi(H_{t-1})) \equiv -k$ on $[0, J^H - \nu]$ where

$$-k < \min \left\{ \frac{\partial \varphi}{\partial K_t}(H_{t-1}, 0) : 0 \leq H_{t-1} \leq J^H - \nu \right\}.$$

Let $p := \min\{\varphi(H_{t-1}, \phi(H_{t-1})) : 0 \leq H_{t-1} \leq J^H - \nu\}$. Applying the Implicit function theorem again, we can find for each $q < p$ a continuous on $[0, J^H - \nu]$ function $\epsilon_q(\cdot)$ such that $\varphi(H_{t-1}, \epsilon_q(H_{t-1})) \equiv q$ on $[0, J^H - \nu]$ and, hence, $\varphi(H_{t-1}, K_t) < q$ for $(H_{t-1}, K_t) \in [0, J^H - \nu] \times \left(\epsilon_q(H_{t-1}), \frac{1}{P_t}(A_t + B_t H_{t-1})\right)$.

Now let q be so small (i.e. big negative) that

$$\begin{aligned} & \sum_{\substack{i=0, \\ i \neq t-1, \\ i \neq t}}^T \left[\beta^i (1 - \zeta^H) \ln \left((1 - \delta) P_i \bar{K}_i + (1 - \tau^L) u_i J^H + r_i \bar{K}_i + M_i \right) + \gamma^H \ln J^H \right] \\ & + \beta^{t-1} (1 - \zeta^H) \ln \left((1 - \delta) P_{t-1} \bar{K}_{t-1} + (1 - \tau^L) u_{t-1} H_{t-1} + r_{t-1} \bar{K}_{t-1} + M_{t-1} \right. \\ & \quad \left. - P_t \epsilon_q(H_{t-1}) \right) + \gamma^H \ln J^H \\ & + \beta^t (1 - \zeta^H) \ln \left((1 - \delta) P_t \epsilon_q(H_{t-1}) + (1 - \tau^L) u_t J^H + r_t \epsilon_q(H_{t-1}) + M_t \right) \\ & \quad + \gamma^H \ln J^H < \Phi(\tilde{H}_0, \tilde{H}_1, \dots, \tilde{H}_T; \tilde{K}_1, \tilde{K}_2, \dots, \tilde{K}_T, \tilde{K}_{T+1}). \end{aligned}$$

Because the function $\epsilon_q(\cdot)$ was constructed for a t fixed in advance (see (25)) from now on we shall denote it by $\epsilon_t(\cdot)$. Hence, if we fix $t \in \{1, 2, \dots, T\}$ and take an arbitrary feasible (for Problem (19)) point $(H_0, H_1, \dots, H_T; K_1, K_2, \dots, K_T, K_{T+1})$ with $K_t \in \left(\epsilon_t(H_{t-1}), \frac{1}{P_t}(A_t + B_t H_{t-1})\right)$, we shall have

$$\Phi(H_0, H_1, \dots, H_T; K_1, K_2, \dots, K_T, K_{T+1})$$

$$< \Phi(\tilde{H}_0, \tilde{H}_1, \dots, \tilde{H}_T; \tilde{K}_1, \tilde{K}_2, \dots, \tilde{K}_T, \tilde{K}_{T+1}).$$

So, the maximum value of Φ can be attained only on the compact subset

$$\prod_{t=0}^T [0, J^H - \nu] \times \prod_{t=1}^T [0, \epsilon_t(H_{t-1})]$$

of the feasible set. Since Φ is continuous, the existence of solution of Problem (19) is proved.

Step 4: Problem (19) has unique solution. To this end it suffices to show that the cost function Φ is strictly concave which can be shown by repeatedly applying Claims 1, 2 and 3. $\square$

C. Properties of the unskilled household's problem.

Proof of Proposition 4. We have denoted the value function of Problem (20) by Ψ . Take any feasible point $(N_t^*, \tilde{N}_t^*)$. It is clear that there exists $\nu > 0$ such that if $J^N - \nu < N_t + \tilde{N}_t < J^N$ then $\Psi(N_t, \tilde{N}_t) < \Psi(N_t^*, \tilde{N}_t^*)$. Hence the maximum of Ψ can be attained only on the compact subset

$$\{(N_t, \tilde{N}_t) \in \mathbf{R}^2 : 0 \leq N_t, 0 \leq \tilde{N}_t, N_t + \tilde{N}_t \leq J^N - \nu\}.$$

of the feasible set. Since Ψ is continuous, the existence of solution of Problem (20) is proved. Since, because of Claim 1, Ψ is strictly concave, the solution is unique. $\square$

D. Derivation of the steady state. In the steady state the variables are expressed as a function of parameter values and based on some additional assumptions for long-term relationships between some of the variables. To this end it is convenient to assume that $Y = 1$ ¹³ in the steady state and express the macroeconomic variables in the steady state as percent of GDP. In particular for the Bulgarian economy we set public debt to be $\bar{B} = 20\%$, which corresponds to the empirical long-term value of this indicator. The fiscal ratios giving social transfers and public services in percent of GDP are respectively set at 15% and 12%, again corresponding to their long-term values in the Bulgarian economy.

The official skilled and unskilled labour H and N are set at 0.0149 and 0.0333, respectively¹⁴.

¹³The values of the variables in the steady state are denoted by removing the time index t .

¹⁴See Appendix A for their derivation.

With respect to the ratio $\frac{\tilde{Y}}{Y}$ of informal to official GDP, it is set at 20% in the steady state. This figure corresponds to around 80% share of consumption in GDP, observed in Bulgaria. This estimate is close to the one reported in [2], but is somewhat lower than around 30% informality calculated in [9]. Finally, the value of the price level P is set at 1 and serves as numerator¹⁵

The transfers Tr , publicly supplied goods and services C^G , public debt B and shadow sector output $\tilde{Y}$ also follow directly from the assumptions on their shares in the official GDP in the steady state.

We can then express the values of the remaining endogenous variables and parameters in the steady state as follows. From equation (22) in the stationary state we have:

$$r = \frac{(1 - \beta(1 - \delta))P}{\beta}$$

Therefore, from the equation about the marginal product of capital in (16) we obtain:

$$K = \frac{\alpha_1 PY}{r}$$

We can then use equation (21) to calibrate the parameter measuring the disutility of work γ^H :

$$\gamma^H = \frac{(1 - \zeta^H)(1 - \tau^L)u}{(1 - \delta)PK + (1 - \tau^L)uH + rK + (1 - \tau^\pi)\pi - (1 - c^N)NTR - PK}(J^H - H)$$

With respect to the unskilled household, by combining equations (23) and (24), we obtain:

$$\tilde{w} = \frac{(1 - \tau^L)w}{(1 - ps\tau^L)}$$

Now we can calculate steady state values of the skilled and unskilled labour remunerations u and w and the informal labor $\tilde{N}$ from the first order conditions (16) and (18).

Similarly to the skilled household, we use equation (23) to calibrate the parameter, accounting for the disutility of work of the unskilled household γ^N :

$$\gamma^N = \frac{(1 - \zeta^H)(1 - \tau^L)w}{(1 - \tau^L)wN + (1 - ps\tau^L)\tilde{w}\tilde{N} + (1 - ps\tau^\pi)\tilde{\pi} + Tr - c^N NTR}(J^N - N - \tilde{N})$$

¹⁵ As per Walras' law if demand equals supply in $k - 1$ markets, then equality on the k^{th} market follows.

Consequently, only $k - 1$ relative prices are required to determine the equilibrium and the general price level needs to be normalized in order to ensure uniqueness of the solution of the system of model equations.

We then use equations (6)–(10) to calculate the steady state values of X , G , as well as to calibrate the values of the parameters c^X , c^{Tr} and c^{CG} .

Finally, the total factor productivity in the official sector z and in the informal production function $\tilde{z}$ are calculated from the respective definitions of the production functions, given in equations (2) and (4).

Tsvetomir Tsachev
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: tsachev@math.bas.bg

Iglika Vassileva
Sofia University
Sofia, Bulgaria
e-mail: iglikav@gmail.com

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