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AN ASPECT OF THE TURNPIKE PROPERTY. LONG TIME HORIZON BEHAVIOR*

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Dedicated to the memory of Asen Dontchev

Abstract. The turnpike phenomenon concerns the structure of the optimal control and the optimal state of dynamic optimal control problems for long time horizons. The focus is regularly placed on the study of the interior of the time interval. Classical turnpike results state how the solution of the dynamic optimal control problems approaches the solution of the corresponding static optimal control problem in the interior of the time interval.

In this paper we look at a new aspect of the turnpike phenomenon. We show that for problems without explicit terminal condition, for large time

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horizons in the last part of the time interval the optimal state approaches a certain limit trajectory that is independent of the terminal time exponentially fast. For large time horizons also the optimal state in the initial part of the time interval approaches exponentially fast a limit state.

1. Introduction. The turnpike phenomenon concerns properties of solutions to dynamic optimal control problems for long time horizons. Usually the studies of the turnpike phenomenon focus on results about the behaviour of the optimal trajectories in the interior of the time interval. These results provide assumptions that imply that for long time horizons, in the interior of the time interval the solutions to the dynamic optimal control problems are approximated by the solutions to the corresponding static optimal control problem.

In this paper we focus on a different aspect, namely the limit trajectories for large time horizons on the initial part and the terminal part of the time interval. We consider dynamic optimal control problems with free terminal state for long time horizons. We show the existence of a limit trajectory on the last part of the time interval. It turns out, that with increasing time horizon T, the terminal arc approaches the limit trajectory that is independent of the time horizon exponentially fast. The corresponding limit states at the time $T - t$ only depend on the distance t to the terminal time T. Hence for long time horizons the optimal state of the dynamic problem in the last part of the time interval approaches a limit trajectory that is independent of the terminal time T. Moreover, also this convergence is exponentially fast.

We also show that on the first part of the time interval $[0, T]$, a limit trajectory that is independent of T does exist such that the optimal state of the dynamic problem approaches exponentially fast the limit trajectory with growing T.

This result is relevant for computations since for large time horizons, the optimal state is very close to the initial limit arc in the first part of the time interval and very close to the terminal limit arc in the last part of the time interval. In the intermediate time interval the optimal state is close to the optimal steady state (the turnpike) that solves the static problem. This structure allows to obtain an excellent approximation of the optimal state for a very large time horizon from the optimal state of a smaller but sufficiently large time horizon by extending the intermediate time interval.

Since there is a substantial amount of literature on the turnpike phenomenon, here we only give a short review. An early reference is [19]. A monograph on the turnpike phenomenon is [25] and an overview on discrete-time and continuous-time turnpike properties in optimal control is given in [5]. The survey

[6] with a particular focus on the control of distributed parameter systems contains also additional references. Measure and integral turnpike properties have been studied in [21]. The turnpike property for systems that are governed by semilinear partial differential equations is studied in [7]. The relation of the turnpike property and the receding-horizon method has been studied in [2]. Manifold turnpikes are studied in [3].

To explain the point that we want to make further we start with a simple example.

Example. Let real numbers $y_d \neq 0$, $T > 0$, $\gamma > 0$ and y_0 be given. Consider the dynamic optimal control problem

$$\begin{cases} \min\limits_{u \in L^2(0,T)} \displaystyle\int_0^T |y(\tau) - y_d|^2 + \gamma\, |u(\tau)|^2 \, d\tau \\ \text{subject to } y(0) = y_0, \;\; y'(t) = y(t) + u(t). \end{cases}$$

The optimal state for the corresponding static optimal control problem

$$\min_{u \in \mathbb{R}} |y - y_d|^2 + \gamma\, |u|^2 \;\text{ subject to } 0 = y + u.$$

is given by $y^{(\sigma)} = \dfrac{1}{1+\gamma}\, y_d$. Define $\omega = \left(1 + \dfrac{1}{\gamma}\right)^{1/2}$.

The optimality conditions imply that the optimal state $\hat{y}_T$ is given by

$$\begin{aligned} \hat{y}_T(t) = y^{(\sigma)} &+ y^{(\sigma)} \frac{1}{\omega - \tanh(\omega T)} \, \frac{\sinh(\omega\, t)}{\cosh(\omega\, T)} \\ &+ \Big[y_0 - y^{(\sigma)}\Big] \, \frac{(\omega+1)\exp(\omega(t-T)) + (\omega-1)\, \exp(\omega(T-t))}{(\omega-1)\exp(\omega\, T) + (\omega+1)\exp(-\omega\, T)}. \end{aligned}$$

Simple calculations show

$$\lim_{T \to \infty} \hat{y}_T(t) = y^{(\sigma)} + \Big[y_0 - y^{(\sigma)}\Big] \exp(-\omega\, t)$$

and

$$\lim_{T \to \infty} \hat{y}_T(T-t) = y^{(\sigma)} + \frac{1}{\omega - 1} y^{(\sigma)} \exp(-\omega t).$$

For $t > 0$ define the initial limit trajectory $y_{init}(t) = y^{(\sigma)} + \Big[y_0 - y^{(\sigma)}\Big] \exp(-\omega\, t)$ that is independent of T. Then $\lim\limits_{T \to \infty} \hat{y}_T(t) = y_{init}(t)$. If $y_0 = y^{(\sigma)}$, we have $y_{init}(t) = y^{(\sigma)}$.

Define the terminal limit trajectory $y_{term}(t) = y^{(\sigma)} + \dfrac{1}{\omega - 1} y^{(\sigma)} \exp(-\omega t)$ that also is independent of T. Then $\lim\limits_{T\to\infty} \hat{y}_T(T-t) = y_{term}(t)$. The limit trajectory y_{term} can only be constant if $y^{(\sigma)} = 0$. We have

$$\lim_{T\to\infty} \hat{y}_T(T) = y_{term}(0).$$

Moreover both limits are reached exponentially fast.

Let $t_0 > 0$ be given. For $t \in (t_0, T - t_0)$ and T sufficiently large we have the inequality

$$|\hat{y}_T(t) - y^{(\sigma)}| \leq 2 \left[\left| y_{init}(0) - y^{(\sigma)} \right| + \left| y_{term}(0) - y^{(\sigma)} \right| \right] \exp(-\omega\, t_0)$$

which implies that in the interior of the time interval, the distance between the static optimal state and the dynamic optimal state decays exponentially fast. Note that by choosing t_0 sufficiently large we can make the upper bound $\exp(-\omega\, t_0)$ arbitrarily small.

This situation indicates that from a practical point of view it makes sense to use a feedback rule in the first part of the time interval to control the system to the optimal static state $y^{(\sigma)}$. If the feedback steers the system close to y_{init} this approach causes almost no loss of optimality compared with the problem for large time horizons.

Define $H_T(t) = \dfrac{1}{2}|\hat{y}_T(t) - y^{(\sigma)}|^2$. We will show that H_T has the meaning of a Lyapunov function for the first part of the time interval where it is decreasing exponentially fast. In contrast to the typical situation with Lyapunov functions, in the last part of the time interval it is increasing. Since $\hat{y}_T'' = \omega^2(\hat{y}_T - y^{(\sigma)})$ we have

$$H_T''(t) = (\hat{y}_T'(t))^2 + (\hat{y}_T(t) - y^{(\sigma)})\hat{y}_T''(t) = (\hat{y}_T'(t))^2 + 2\,\omega^2\, H_T(t).$$

Hence H_T is convex. We have $H_T'(0) = (y_0 - y^{(\sigma)})\hat{y}_T'(0)$ and

$$\lim_{T\to\infty} \hat{y}_T'(0) = -\omega(y_0 - y^{(\sigma)}) = y_{init}'(0).$$

If T is sufficiently large and $y_0 \neq y^{(\sigma)}$ we obtain $H_T'(0) < 0$. This implies that in the first part of $[0,\, T]$, H_T is strictly decreasing.

If $t_T^* > 0$ is such that $\hat{y}_T'(t_T^*) = 0$, we have $H_T'(t_T^*) = 0$. At such a point $t_T^* > 0$, the function $H_T \geq 0$ attains its minimal value. For $t > t_T^*$, the value of $H_T(t)$ can increase again and if $y^{(\sigma)} \neq 0$ this is what happens in the last part of

the time-interval where $t \mapsto \hat{y}_T(T-t)$ approaches $t \mapsto y_{term}(t)$ for $T \to \infty$. In fact, since

$$\lim_{T\to\infty} \hat{y}'_T(T) = \frac{\omega}{\omega - 1} y^{(\sigma)} = -y'_{term}(0)$$

in this case we have

$$\lim_{T\to\infty} H'_T(T) = \lim_{T\to\infty} (\hat{y}_T(T) - y^{(\sigma)})\, \hat{y}'_T(T) = -(y_{term}(0) - y^{(\sigma)}) y'_{term}(0) > 0.$$

The aim of this paper is to show that the situation that occurs in our example can also be found in optimal control problems with systems that are governed by partial differential equations.

The structure of the paper. In Section 2 we present another example to illustrate the aspect of the turnpike phenomenon that we investigate with a system that is governed by the transport equation. In Section 3 we define a general optimal control problem that we analyze. We consider a system that is governed by a semi–group of contractions. We define a linear-quadratic optimal control problem.

In Section 4 we derive the optimality systems both for the dynamic optimal control problem and the corresponding static optimal control problem. Then we consider the difference between the dynamic and the static optimal solutions and derive a system with an initial condition and a terminal condition that is satisfied by this difference.

In Section 5 we show the existence of two limit trajectories for the boundaries of the time interval. We have initial limit trajectories for fixed times $t > 0$ and terminal limit trajectories for times $T-t$ with a fixed distance to the terminal time T. We derive representations of the long-time horizon limit terminal state, the long-time horizon limit trajectory for the last part of the time interval and the initial limit trajectory. We provide a representation in terms of the optimal adjoint state for the static problem and show the convergence towards the limit trajectory is exponentially fast.

In Section 6 we present a finite dimensional example and an example with the wave equation to illustrate our findings. At the end of the paper in Section 7 we point out possible directions of future research.

2. An example with the transport equation. In order to illustrate further the aspect of the turnpike phenomenon that we want to study, we present another example with a system that is governed by a hyperpoblic partial differential equation.

Let $L > 0$, $T > L$, $\gamma > 0$ and $y_d \in \mathbb{R}$ be given. Let $Q = [0,\, T] \times [0,\, L]$. Let an initial state $y_0 \in C([0,\, L])$ be given. Consider the optimal control problem

$$\begin{cases} \min\limits_{u \in L^2(0,T)} \displaystyle\int_0^T \int_0^L |y(t,x) - y_d|^2 \, dx \, dt + \gamma \int_0^T |u(t)|^2 \, dt \\ \text{subject to } y(0,\, x) = y_0(x) \text{ for } x \in [0,\, L],\ y(t,\, 0) = u(t) \text{ for } t \in [0, T] \text{ a.e.,} \\ \text{and } y_t + y_x = 0 \text{ on } Q. \end{cases}$$

The state on the triangle $G = \{(t, x); x \in [0,\, L], 0 \le t \le x\}$ depends on the initial state only and is not influenced by the control. Hence the contribution from G to the integral over Q in the objective functions does not play a role for the optimal control.

The state is constant on the characteristic lines that have the form $t = x + c$ for a real constant c. The values of the state on the different characteristic lines corresponding to different values of c are independent of each other. Thus we can decompose the problem in the family of optimization problems corresponding to the different characteristic lines that appear on Q.

For a constant $c \in [0,\, T]$ let $L_{term}(c) \in [0,\, L]$ be such that the characteristic curve corresponding to c is contained in Q for all $x \in [0, L_{term}(c)]$. If $c \le T - L$ we have $L_{term}(c) = L$ and if $c > T - L$ we have $L_{term}(c) = T - c$.

The optimization problem for the characteristic curve corresponding to c is

$$\begin{cases} \min\limits_{u(c) \in \mathbb{R}} \displaystyle\int_0^{L_{term}(c)} |y(s + c, s) - y_d|^2 \, ds + \gamma |u(c)|^2 \\ \text{subject to } y(c,\, 0) = u(c) \text{ and } y(s + c, s) = y(c,\, 0) \text{ for all } s \in [0, L_{term}(c)]. \end{cases}$$

Since for all $s \in [0, L_{term}(c)]$ we have $y(s + c, s) = u(c)$ the solution is equal to the minimum of the polynomial

$$p_c(\nu) = L_{term}(c)\, |\nu - y_d|^2 + \gamma\, |\nu|^2$$

that is given by

$$\hat{u}_T(c) = \frac{1}{1 + \frac{\gamma}{L_{term}(c)}} y_d.$$

Hence for all $t \in [0, T - L]$ we have the optimal control

$$\hat{u}_T(t) = \frac{1}{1 + \frac{\gamma}{L}} y_d$$

and the optimal state

$$\hat{y}_T(x+t,\, x) = \hat{u}_T(t). \tag{1}$$

So we see that in this example for all $t \in [0, T-L]$ the state $\hat{y}_T(t, \cdot)$ is independent of T. Hence for all $t \in (L,\, T-L]$ and $z \in [0, L]$ we have

$$y_{init}(t,\, z) = \lim_{T\to\infty} \hat{y}_T(t, z) = \frac{1}{1+\frac{\gamma}{L}} y_d.$$

We call $y_{init}(t,\, z)$ the initial limit trajectory.

For $t \in (T-L, T]$ the optimal control is

$$\hat{u}_T(t) = \frac{1}{1+\frac{\gamma}{T-t}} y_d$$

and the optimal state is given by (1). So we see that $\hat{u}_T(t)$ only depends on $T-t$. For $t \in (0, L)$ and $z \in [0,\, t]$ when taking the limit $T \to \infty$ we have

$$y_{term}(t,\, z) = \lim_{T\to\infty} \hat{y}_T(T-t, z) = \frac{1}{1+\frac{\gamma}{t}} y_d.$$

We call $y_{term}(t,\, z)$ the terminal limit trajectory. Moreover, we have

$$y_{term}(0,\, 0) = \lim_{T\to\infty} \hat{y}_T(T, 0) = 0.$$

The static optimal control problem is

$$\begin{cases} \min\limits_{u\in\mathbb{R}} \displaystyle\int_0^L |y(x) - y_d|^2 \, dx + \gamma |u|^2 \\ \text{subject to } y(0) = u \text{ and } y_x = 0 \text{ on } [0, L]. \end{cases}$$

Hence we obtain the static optimal state

$$y^{(\sigma)} = \frac{1}{1+\frac{\gamma}{L}} y_d$$

and the optimal control $u^{(\sigma)} = y^{(\sigma)}$. Thus for all $t \in (L,\, T-L]$ and $z \in [0, L]$ for the limit state we have

$$y_{init}(t,\, z) = y^{(\sigma)}.$$

The state for $t \in [0, L]$ is also independent of T since the values of y on the triangle G are independent of T.

So we see that in this example, we obtain explicit representations of the optimal state that show that on the interior time interval $(L,\, T-L)$ we have $\hat{y}_T(t,x) = y^{(\sigma)}$ and $\hat{u}_T(t) = u^{(\sigma)}$. Thus in this example, the optimal static state is reached exactly in finite time. This is similar as in the finite-time turnpike phenomenon that is described in [12]. However, in [12] this situation is enforced by non-smooth tracking terms in the objective functional.

The structure of the solution in the example is similar as in the numerical examples in [9] where the boundary control of a linear hyperbolic 2×2 system with space dimension one is studied. A related application is studied in [14], namely optimal control problems with the two–dimensional transport equation that are used for optimal treatment planning in radiotherapy.

3. The dynamic optimal control problem. In this section, we present an abstract setting where it can be shown that initial and terminal limit trajectories of the type that we have discussed above exist. Note that the examples that we have presented above to illustrate the phenomenon of initial limit trajectories and terminal limit trajectories for large time horizons are not special cases of the framework that we consider below. In Section 6 we will present examples that are covered by this framework.

Let U and X denote complex Hilbert spaces and $\mathbb{T}_t$ a strongly continuous semi-group of contractions on X with generator A. Let $B \in \mathcal{L}(U,\, X_{-1})$. Here we denote by X_{-1} the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta\, I - A)^{-1} z\|$ where β is an element of the resolvent set $\rho(A) \neq \emptyset$ and $\mathcal{L}$ denotes the space of bounded linear operators from U to X_{-1}. Our setting is as in [22], Chapter 4. We refer also to the monographs [17] and [18]. We consider a system that is governed by the differential equation

$$z'(t) = A\, z(t) + B\, u(t) \tag{2}$$

where B is an admissible control operator that is defined in U and controls $u \in L^2_{loc}([0,\, \infty), U)$. Then for all $t \geq 0$ we can represent the state in the form

$$z(t) = z(0) + \int_0^t [Az(s) + Bu(s)]\, ds$$

(see Proposition 4.2.5 in [22]). We have $z \in C([0,\, \infty); X)$. We can also write

$$z(t) = \mathbb{T}_t z(0) + \int_0^t \mathbb{T}_{t-s} B u(s)\, ds.$$

Note that since B is an admissible control operator, for $t \geq 0$ the operator

$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} B u(s)\, ds$$

is bounded and the adjoint semigroup $\mathbb{T}_t^*$ is also a strongly continuous semigroup on X that is generated by A^*. We assume that A is *skew-adjoint*, that is for all v, w in the domain of A we have

$$\langle Aw,\, v\rangle = \langle w,\, -Av\rangle. \tag{3}$$

We also assume that $0 \in \rho(A)$ as in Proposition 3.7.6. in [22]. Proposition 3.7.2 in [22] states that (3) is equivalent to the statement that both A and $-A$ are m-dissipative. Hence the Lumer-Phillips Theorem implies that A is the generator of a semi-group of contractions.

Let an initial state $z_0 \in X$ and a desired state $z_d \in X$ be given. Let $T > 0$ be given. For $u \in L^2([0,\, T], U)$ we define the objective functional

$$J_T(u) = \int_0^T \|z(s) - z_d\|_X^2 + \|u(s)\|_U^2\, ds$$

where the state z is defined as the solution of the initial value problem

$$z(0) = z_0,\; z'(t) = Az(t) + Bu(t). \tag{4}$$

We consider the dynamic optimal control problem

$$\mathbf{P}(T) : \min_{u \in L^2([0,\, T], U)} J_T(u). \tag{5}$$

Note that since the objective function is strongly convex, $\mathbf{P}(T)$ can have at most one solution. Moreover, our assumptions allow to prove the existence of a solution of $\mathbf{P}(T)$ using the direct method of the calculus of variations. This is done with the classical approach to consider a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ of controls with $u_n \in L^2([0,\, T], U)$. Due to the growth of the objective function this sequence is bounded in this space. Due to the convexity of the problem this implies the existence of a minimizer.

4. Optimality conditions. In this section we study the necessary optimality conditions for the dynamic optimal control problem $\mathbf{P}(T)$ and the corresponding static optimal control problem. We also derive a differential equation for the difference of the dynamic optimal state/control pair and the static optimal state/control pair. This system is completed by an initial condition for the state and a terminal condition for the adjoint state.

4.1. Necessary optimality conditions for the dynamic optimal control problem. First we derive the necessary optimality conditions for the dynamic optimal control problem $\mathbf{P}(T)$. We introduce a variation δu of the control and the corresponding variation δz of the generated state. Then we have the initial condition $\delta z(0) = 0$ and the differential equation $\delta z' = A\,\delta z + B\,\delta u$. We introduce an adjoint state μ with the same regularity as z and obtain the optimality system

$$z(0) = z_0,\ \mu(T) = 0 \tag{6}$$

$$z' = Az + Bu \tag{7}$$

$$\mu' = -A^*\mu + z - z_d \tag{8}$$

$$u = B^*\mu. \tag{9}$$

We state the necessary optimality conditions in the following Lemma:

Lemma 1. *The solution of* $\boldsymbol{P}(T)$ *satisfies the necessary and sufficient optimality conditions* (6)–(9) *with the adjoint state* μ *that satisfies* (8) *in the sense*

$$\mu(t) = \int_T^t [A\,\mu(s) + (z(s) - z_d)]\,ds.$$

P r o o f. For all control variations δu, for the objective value we have

$$\begin{aligned} J_T(u+\delta u) &= \int_0^T \|z(s) + \delta z(s) - z_d\|_X^2 + \|u(s) + \delta u(s)\|_U^2\,ds \\ &= J_T(u) + 2\int_0^T \langle z(s) - z_d,\, \delta z(s)\rangle_X + \langle u,\, \delta u\rangle_U\,ds \\ &+ \int_0^T \|\delta z(s)\|_X^2 + \|\delta u(s)\|_U^2 \\ &\geq J_T(u) + 2\int_0^T \langle z(s) - z_d,\, \delta z(s)\rangle_X + \langle u,\, \delta u\rangle_U\,ds. \end{aligned}$$

Due to (7), our variations δu of the control generates a variation δz of the state in the sense that the function δz is given by

$$\delta z(t) = \int_0^t [A\delta z(s) + B\delta u(s)]\,ds.$$

Hence we have

$$J_T(u+\delta u) \geq J_T(u)$$

$$+ 2\int_0^T \left\langle z(t) - z_d,\ \int_0^t [A\delta z(s) + B\delta u(s)]\, ds \right\rangle_X + \langle u,\ \delta u\rangle_U\, dt.$$

We use integration by parts in the weak form that for functions $g,\ h \in C([0,\ T],\ X)$ we have

$$\int_0^T \left\langle g(t),\ \int_0^t h(s)\, ds \right\rangle_X dt = \int_0^T \left\langle \int_t^T g(s)\, ds,\ h(t) \right\rangle_X dt. \tag{10}$$

Using (10) we obtain the inequality

$$J_T(u+\delta u) \geq J_T(u) + 2\int_0^T \left\langle \int_t^T z(s) - z_d\, ds,\ [A\delta z(t) + B\delta u(t)] \right\rangle_X + \langle u,\ \delta u\rangle_U\, dt.$$

For the adjoint state, (8) is equivalent to

$$\mu(t) = -\int_T^t [A^*\mu(s) - (z(s) - z_d)]\, ds = \int_t^T [A^*\mu(s) - (z(s) - z_d)]\, ds.$$

This in turn is equivalent to

$$\int_t^T [z(s) - z_d]\, ds = -\mu(t) + \int_t^T A^*\mu(s)\, ds. \tag{11}$$

Now (11) yields

$$J_T(u+\delta u) \geq J_T(u) + 2\int_0^T \left\langle -\mu(t) + \int_t^T A^*\mu(s)\, ds,\ [A\delta z(t) + B\delta u(t)] \right\rangle_X + \langle u,\ \delta u\rangle_U\, dt.$$

Hence we obtain the inequality

$$\begin{aligned} J_T(u+\delta u) - J_T(u) &\geq 2\int_0^T \left\langle -\mu(t) + \int_t^T A^*\mu(s)\, ds,\ A\delta z(t) \right\rangle_X \\ &\quad + \left\langle \int_t^T A^*\mu(s)\, ds,\ B\delta u(t) \right\rangle_X + \langle u - B^*\mu,\ \delta u\rangle_U\, dt \\ &= 2\int_0^T \langle -\mu(t),\ A\delta z(t)\rangle_X \end{aligned}$$

$$
\begin{aligned}
&\quad + \left\langle \int_t^T A^*\mu(s)\,ds,\ A\delta z(t) + B\delta u(t) \right\rangle_X + \langle u - B^*\mu,\ \delta u\rangle_U\,dt \\
&= 2\int_0^T \langle -\mu(t),\ A\delta z(t)\rangle_X \\
&\qquad + \left\langle A^*\mu(t),\ \int_0^t A\delta z(s) + B\delta u(s)\,ds \right\rangle_X + \langle u - B^*\mu,\ \delta u\rangle_U\,dt \\
&= 2\int_0^T \langle -\mu(t),\ A\delta z(t)\rangle_X + \langle A^*\mu(t),\ \delta z(t)\rangle_X + \langle u - B^*\mu,\ \delta u\rangle_U\,dt \\
&= 2\int_0^T \langle u - B^*\mu,\ \delta u\rangle_U\,dt.
\end{aligned}
$$

Whence the inequality

$$J_T(u+\delta u) - J_T(u) \geq 0 \tag{12}$$

can only hold for all control variations δu if (9) holds.

Thus we obtain the necessary optimality conditions.

If the optimality system holds, we also obtain the inequality (12), hence the optimality conditions are also sufficient. □

Note that the optimality system is already a standard result, see for example [18] for a slightly different setting.

4.2. Necessary optimality conditions for the static optimal control problem. The static optimal control problem corresponding to $\mathbf{P}(T)$ is obtained by cancelling the initial condition and the time dependence. We have the static optimal control problem

$$\mathbf{S}: \min_{u\in U} \|z - z_d\|_X^2 + \|u\|_U^2 \text{ subject to } 0 = A\,z + B\,u. \tag{13}$$

Again the existence of a solution follows with the direct method of the calculus of variations. The feasible set is non empty since the control $0 \in U$ is feasible. Hence the minimal objective value is less then or equal to $\|z_d\|_X^2$. Thus every minimizing sequence $(u_n)_n$ of controls $u_n \in U$ is bounded, we have $\|u_n\|_U^2 \leq \|z_d\|_X^2$. Due to the convexity of the problem this implies the existence of a minimizer.

The uniqueness follows from the strict convexity of the objective functional. We denote the unique solution of $\mathbf{S}$ by $(z^{(\sigma)},\ u^{(\sigma)})$.

We introduce a variation δu of the control and the corresponding variation δz of the generated state. Then we have $0 = A\,\delta z + B\,\delta u$. We introduce an adjoint

state $\mu \in X$ and obtain the optimality system

$$0 = Az + Bu \tag{14}$$

$$0 = -A^*\mu + z - z_d \tag{15}$$

$$u = B^*\mu. \tag{16}$$

We state the necessary optimality conditions in the following Lemma:

Lemma 2. *The solution of* $\boldsymbol{S}$ *satisfies the necessary and sufficient optimality conditions* (14)–(16) *with the adjoint state* μ.

P r o o f. For all control variations δu, for the objective value we have

$$\begin{aligned}
&\|z+\delta z - z_d\|_X^2 + \|u + \delta u\|_U^2 \\
&\quad = \|z - z_d\|_X^2 + \|u\|_U^2 + 2\langle z - z_d,\, \delta z\rangle_X + 2\langle u,\, \delta u\rangle_U + \|\delta z\|_X^2 + \|\delta u\|_U^2 \\
&\quad \geq \|z - z_d\|_X^2 + \|u\|_U^2 + 2\langle z - z_d,\, \delta z\rangle_X + 2\langle u,\, \delta u\rangle_U.
\end{aligned}$$

Hence we obtain the inequality

$$\begin{aligned}
&\|z+\delta z - z_d\|_X^2 + \|u + \delta u\|_U^2 - \|z - z_d\|_X^2 - \|u\|_U^2 \\
&\quad \geq 2\langle z - z_d,\, \delta z\rangle_X + 2\langle u,\, \delta u\rangle_U \\
&\quad = 2\langle z - z_d,\, \delta z\rangle_X + 2\langle u - B^*\mu + B^*\mu,\, \delta u\rangle_U.
\end{aligned}$$

Since $0 \in \rho(A)$, due to (14) we have $\delta z = -A^{-1}B\delta u$. This yields

$$\begin{aligned}
&\|z+\delta z - z_d\|_X^2 + \|u + \delta u\|_U^2 - \|z - z_d\|_X^2 - \|u\|_U^2 \\
&\quad \geq 2\langle z - z_d,\, -A^{-1}B\delta u\rangle_X + 2\langle B^*\mu,\, \delta u\rangle_U + 2\langle u - B^*\mu,\, \delta u\rangle_U \\
&\quad = 2\langle -B^*(A^{-1})^*(z - z_d),\, \delta u\rangle_U + 2\langle B^*\mu,\, \delta u\rangle_U + 2\langle u - B^*\mu,\, \delta u\rangle_U \\
&\quad = 2\langle -B^*(A^*)^{-1}(z - z_d) + B^*\mu,\, \delta u\rangle_U + 2\langle u - B^*\mu,\, \delta u\rangle_U \\
&\quad = 2\langle B^*\left(-(A^*)^{-1}(z - z_d) + \mu\right),\, \delta u\rangle_U + 2\langle u - B^*\mu,\, \delta u\rangle_U.
\end{aligned}$$

For an adjoint state μ that is defined by (15) we obtain the inequality

$$\|z + \delta z - z_d\|_X^2 + \|u + \delta u\|_U^2 - \|z - z_d\|_X^2 - \|u\|_U^2 \geq 2\langle u - B^*\mu,\, \delta u\rangle_U.$$

Hence the inequality

$$\|z + \delta z - z_d\|_X^2 + \|u + \delta u\|_U^2 \geq \|z - z_d\|_X^2 + \|u\|_U^2 \tag{17}$$

can only hold for all control variations δu, if (16) holds.

Note that if the optimality system (14)–(16) holds, we also obtain the inequality (17), hence the optimality conditions are also sufficient. □

4.3. A system for the difference between the static and the dynamic solution. Now we derive a system for $\hat{z}_T - z^{(\sigma)}$ and $\hat{\mu}_T - \mu^{(\sigma)}$ where $(\hat{z}_T, \hat{\mu}_T)$ is the optimal pair of state and adjoint state and $\hat{u}_T$ denotes the optimal control for $\mathbf{P}(T)$ and $(z^{(\sigma)}, \mu^{(\sigma)})$ is the optimal state and optimal adjoint state for $\mathbf{S}$ and $u^{(\sigma)}$ denotes the optimal control for $\mathbf{S}$. Due to (6)–(9) and (14)–(16) we have

$$(\hat{z}_T - z^{(\sigma)})(0) = z_0 - z^{(\sigma)},\ \hat{\mu}_T(T) - \mu^{(\sigma)} = -\mu^{(\sigma)}, \tag{18}$$

$$(\hat{z}_T - z^{(\sigma)})' = A(\hat{z}_T - z^{(\sigma)}) + B(\hat{u}_T - u^{(\sigma)}), \tag{19}$$

$$(\hat{\mu}_T - \mu^{(\sigma)})' = -A^*(\hat{\mu}_T - \mu^{(\sigma)}) + (\hat{z}_T - z^{(\sigma)}), \tag{20}$$

$$\hat{u}_T - u^{(\sigma)} = B^*(\hat{\mu}_T - \mu^{(\sigma)}). \tag{21}$$

Due to (3), we can derive an explicit representation of the general solution of the differential equation (19), (20) that takes into account (21). in order to derive some convergence results.

Lemma 3. *Assume that A and $B\,B^*$ commute. Due to our assumptions, the operator*

$$\Lambda = (BB^*)^{1/2} \tag{22}$$

is well–defined as the square root of a positive self-adjoint operator (see [1], [20], [23]*). For the solution of*

$$(z - z^{(\sigma)})' = A(z - z^{(\sigma)}) + \Lambda^2(\mu - \mu^{(\sigma)}), \tag{23}$$

$$(\mu - \mu^{(\sigma)})' = (z - z^{(\sigma)}) - A^*(\mu - \mu^{(\sigma)}) \tag{24}$$

we have

$$(z - z^{(\sigma)})(t) = \frac{1}{2}\left[\exp((A + \Lambda)\,t)\,l_+ + \exp((A - \Lambda)\,t)\,l_-\right], \tag{25}$$

$$\Lambda(\mu - \mu^{(\sigma)})(t) = \frac{1}{2}\left[\exp((A + \Lambda)\,t)\,l_+ - \exp((A - \Lambda)\,t)\,l_-\right] \tag{26}$$

where l_+ and $l_- \in X$ are uniquely determined by (18).

We use the notation $\hat{z}_T$ for the optimal state of $\boldsymbol{P}(T)$. If

$$B\,B^* > 0 \tag{27}$$

we have

$$\lim_{T\to\infty} \hat{z}_T(T) = z^{(\sigma)} - \Lambda\,\mu^{(\sigma)}. \tag{28}$$

P r o o f. Let $l_+ \in X$ and $l_- \in X$ be given. For $t \in [0,\, T]$, we define

$$\alpha(t) = \exp((A+\Lambda)\,t)\,l_+,\ \beta(t) = \exp((A-\Lambda)\,t)\,l_-.$$

Then we have

$$\partial_t \alpha(t) = (A+\Lambda)\exp((A+\Lambda)t)\,l_+ = (A+\Lambda)\,\alpha(t).$$

Our assumptions imply that Λ and A commute. Hence (23) and (24) yield

$$\Big((z - z^{(\sigma)}) + \Lambda(\mu - \mu^{(\sigma)})\Big)' = (A+\Lambda)\Big((z - z^{(\sigma)}) + \Lambda(\mu - \mu^{(\sigma)})\Big).$$

Thus for an appropriately chosen $l_+ \in X$ we have

$$(z - z^{(\sigma)}) + \Lambda(\mu - \mu^{(\sigma)}) = \alpha(t). \tag{29}$$

From (23) and (24) we also obtain

$$\Big((z - z^{(\sigma)}) - \Lambda(\mu - \mu^{(\sigma)})\Big)' = (A-\Lambda)\Big((z - z^{(\sigma)}) - \Lambda(\mu - \mu^{(\sigma)})\Big).$$

Thus for an appropriately chosen $l_- \in X$ we have

$$(z - z^{(\sigma)}) - \Lambda(\mu - \mu^{(\sigma)}) = \beta(t). \tag{30}$$

Now (29) and (30) yield (25) and (26).

The initial and terminal conditions (18) yield the following system of linear equations for $(l_+,\, l_-)$:

$$l_+ + l_- = 2(z_0 - z^{(\sigma)}), \tag{31}$$

$$\exp((A+\Lambda)T)\,l_+ - \exp((A-\Lambda)T)\,l_- = -2\,\Lambda\mu^{(\sigma)}. \tag{32}$$

Equation (32) implies

$$l_+ - \exp(-2\Lambda T)l_- = -2\,\Lambda\exp(-(A+\Lambda)T)\mu^{(\sigma)}.$$

With (31) this yields

$$(I + \exp(-2\Lambda T))l_- = 2(z_0 - z^{(\sigma)}) + 2\,\Lambda\exp(-(A+\Lambda)T)\mu^{(\sigma)}.$$

Thus we obtain

$$l_- = (I + \exp(-2\Lambda T))^{-1} \left[2(z_0 - z^{(\sigma)}) + 2\,\Lambda \exp(-(A+\Lambda)T)\mu^{(\sigma)} \right]$$

and similarly

$$l_+ = (I + \exp(2\Lambda T))^{-1} \left[2(z_0 - z^{(\sigma)}) - 2\,\Lambda \exp(-(A-\Lambda)T)\mu^{(\sigma)} \right].$$

In the sequel, we use the notation $l_\pm(T)$ for $l_\pm$ as a function of T because we are interested in the convergence for $T \to \infty$. On account of (3) and (27) we have $\lim_{T\to\infty} l_-(T) = 2(z_0 - z^{(\sigma)})$. With (31) this implies $\lim_{T\to\infty} l_+(T) = 0$.

We introduce the notation $\hat{z}_T$ for the optimal state corresponding to the time horizon T. For $t \in [0,\, T]$ due to (25) we have

$$\hat{z}_T(t) - z^{(\sigma)} = \frac{1}{2} \left[\exp((A+\Lambda)\,t)\, l_+(T) + \exp((A-\Lambda)\,t)\, l_-(T) \right]. \tag{33}$$

For $t \in [0,\, T]$ we set

$$\hat{z}_T^{(+)}(t) = \frac{1}{2} \exp((A+\Lambda)\,t)\, l_+(T),\ \hat{z}_T^{(-)}(t) = \frac{1}{2} \exp((A-\Lambda)t)\, l_-(T). \tag{34}$$

Then we have

$$\hat{z}_T(T) - z^{(\sigma)} = \hat{z}_T^{(+)}(T) + \hat{z}_T^{(-)}(T). \tag{35}$$

We have

$$l_-(T) = (I + \exp(-2\Lambda T))^{-1} \left[2(z_0 - z^{(\sigma)}) + 2\,\Lambda \exp(-(A+\Lambda)T)\mu^{(\sigma)} \right].$$

Assumption (27) implies $\Lambda > 0$. This implies

$$\lim_{T\to\infty} (I + \exp(-2\Lambda T))\, l_-(T) = 2(z_0 - z^{(\sigma)}).$$

It follows that

$$\lim_{T\to\infty} l_-(T) = 2(z_0 - z^{(\sigma)}).$$

Since we have

$$\hat{z}_T^{(-)}(T) = \frac{1}{2} \exp((A-\Lambda)t)\, l_-(T)$$

therefore we have $\lim_{T\to\infty} \hat{z}_T^{(-)}(T) = 0$. Define

$$\mathcal{N}(T) = (I + \exp(2\,\Lambda\, T))^{-1} = (I + \exp(-2\,\Lambda\, T))^{-1} \exp(-2\,\Lambda\, T).$$

We also use the notation

$$l_+^{(1)}(T) = \mathcal{N}(T)(z_0 - z^{(\sigma)}), \tag{36}$$

$$l_+^{(2)}(T) = \mathcal{N}(T)\Lambda \exp(-(A-\Lambda)T)\mu^{(\sigma)}. \tag{37}$$

We have

$$l_+(T) = 2l_+^{(1)}(T) - 2l_+^{(2)}(T) \tag{38}$$

and

$$z_T^{(+)}(T) = \exp((A+\Lambda)T)\, l_+^{(1)}(T) - \exp((A+\Lambda)T)\, l_+^{(2)}(T).$$

We have

$$\exp((A+\Lambda)T)\,\mathcal{N}(T) = \exp((A-\Lambda)T)\,(I + \exp(-2\,\Lambda\,T))^{-1}.$$

Hence for the limit we have

$$\lim_{T\to\infty} \exp((A+\Lambda)T)\, l_+^{(1)}(T) = 0.$$

For the second term in our representation of $z_T^{(+)}(T)$ we have

$$\exp((A+\Lambda)T)\, l_+^{(2)}(T) = \Lambda \exp(2\Lambda\,T)\,\mathcal{N}(T)\mu^{(\sigma)} = \Lambda\,(I + \exp(-2\,\Lambda\,T))^{-1}\mu^{(\sigma)}.$$

This yields the limit

$$\lim_{T\to\infty} \exp((A+\Lambda)T)\, l_+^{(2)}(T) = \Lambda\,\mu^{(\sigma)}$$

which implies

$$\lim_{T\to\infty} \hat{z}_T(T) - z^{(\sigma)} = -\Lambda\,\mu^{(\sigma)}.$$

Now the assertion follows. □

5. The existence of two limit trajectories for the boundaries of the time interval. Now we state our result on the limit trajectory on the last part of the time interval. Let $t_0 > 0$ be fixed and $T > t_0$, We show that more generally than in Lemma 3, in the last part of the time interval $[0, T]$ (that is in $[T - t_0, T]$), the optimal trajectory approaches exponentially fast a limit-trajectory that is independent of the time horizon and where the value at the time $T - t$ only depends on t.

Theorem 1. *Assume that A and $B\,B^*$ commute and that* (27) *holds. Let Λ be defined as in* (22). *Let $m \in (0,\,1)$ denote a constant such that*

$$\|\Lambda\, x\|_X \geq m\|x\|_X. \tag{39}$$

Assume that

$$T \geq \frac{1}{m}. \tag{40}$$

For $s \in [0,T]$, define the limit trajectory

$$\zeta(s) = z^{(\sigma)} - \Lambda\, \exp(-(A+\Lambda)s)\, \mu^{(\sigma)}.$$

For $t_0 > 0$ and all $t \in [T-t_0,\,T]$ we have

$$\|\hat{z}_T(t) - \zeta(T-t)\|_X \leq 2\,\frac{1+\exp(mt_0)}{\exp(m\,T)}\,\left[\|z_0 - z^{(\sigma)}\|_X + \|\Lambda\,\mu^{(\sigma)}\|_X\right]. \tag{41}$$

This means that for $t \in [T-t_0, T]$ the optimal state $\hat{z}_T(t)$ approaches the limit trajectory $\zeta(T-t)$ exponentially fast with respect to T. Note that the point $\zeta(T-t)$ only depends on the distance to the terminal time T.

P r o o f. We use the same notation as in the proof of Lemma 3, in particular let $l_+(T)$ be defined as in (38). This implies the equation

$$l_+(T) = 2\mathcal{N}(T)(z_0 - z^{(\sigma)}) + 2\mathcal{N}(T)\,\exp(\Lambda\,T)\,\Lambda\mu^{(\sigma)}.$$

For the operator norm of $\mathcal{N}(T)$ we have

$$\|\mathcal{N}(T)\| \leq \frac{\exp(-2\,m\,T)}{1-\exp(-2\,m\,T)},$$

this follows e.g. with Proposition 3.8.5. from [22]. Moreover,

$$\|\mathcal{N}(T)\,\exp(\Lambda\,T)\| = \|(I+\exp(-2\,\Lambda\,T))^{-1}\,\exp(-\Lambda\,T)\| \leq \frac{\exp(-m\,T)}{1-\exp(-2m\,T)}.$$

Thus we obtain the inequality

$$\|l_+(T)\| \leq \frac{\exp(-m\,T)}{1-\exp(-2m\,T)}\,\left[\|(z_0 - z^{(\sigma)})\|_X + \|\Lambda\mu^{(\sigma)}\|_X\right]. \tag{42}$$

Due to (31) this implies

$$\|l_-(T)\|_X \leq \|l_+(T)\|_X + 2\,\|(z_0 - z^{(\sigma)})\|_X \tag{43}$$

$$\leq 2\,\|(z_0 - z^{(\sigma)})\|_X + \frac{\exp(-m\,T)}{1-\exp(-2m\,T)}\left[\|(z_0 - z^{(\sigma)})\|_X + \|\Lambda\mu^{(\sigma)}\|_X\right].$$

Note that (40) implies $\dfrac{\exp(-m\,T)}{1-\exp(-2m\,T)} \leq 1$. Similar as in (35), for all $t \in [0,T]$ we have

$$\hat{z}_T(t) - z^{(\sigma)} = \hat{z}_T^{(+)}(t) + \hat{z}_T^{(-)}(t). \tag{44}$$

For all $t \in [T-t_0,\, T]$ have equation

$$\begin{aligned}\|\hat{z}_T^{(-)}(t)\|_X &\leq \frac{1}{2}\exp(-m(T-t_0))\|l_-(T)\|_X \\ &\leq 2\,\exp(-m(T-t_0))\left[\|z_0 - z^{(\sigma)}\|_X + \|\Lambda\mu^{(\sigma)}\|_X\right].\end{aligned} \tag{45}$$

We have

$$z_T^{(+)}(t) = \exp((A+\Lambda)t)\,l_+^{(1)}(T) - \exp((A+\Lambda)t)\,l_+^{(2)}(T). \tag{46}$$

Thus for the first term of $z_T^{(+)}(t)$ we obtain

$$\exp((A+\Lambda)t)\,l_+^{(1)}(T) = \exp(At)\,\exp(\Lambda t)\mathcal{N}(T)(z_0 - z^{(\sigma)}).$$

Hence for all $t \in [0,T]$ we have the inequality

$$\|\exp((A+\Lambda)t)\,l_+^{(1)}(T)\|_X \leq 2\,\exp(-m\,T)\|z_0 - z^{(\sigma)}\|_X. \tag{47}$$

Now we consider the next term in our representation of $z_T^{(+)}(t)$. In view of (46), for the second term in our representation of $z_T^{(+)}(t)$ we have

$$\exp((A+\Lambda)t)\,l_+^{(2)}(T) = \Lambda\,\exp(A(t-T))\,\exp(\Lambda(t-T))\,\exp(2\,\Lambda T)\mathcal{N}(T)\mu^{(\sigma)}. \tag{48}$$

Due to the Neumann series for the corresponding operator norm we have

$$\|\exp(2\,\Lambda T)\mathcal{N}(T) - I\| \leq \frac{\exp(-2mT)}{1-\exp(-2m\,T)}.$$

Hence for all $t \in [0,T]$ we have the inequality

$$\begin{aligned}\|-\exp((A+\Lambda)t)\,l_+^{(2)}(T) + \Lambda\,\exp((A+\Lambda)(t-T))\mu^{(\sigma)}\|_X \\ \leq \frac{\exp(-2mT)}{1-\exp(-2m\,T)}\|\Lambda\,\mu^{(\sigma)}\|_X.\end{aligned}$$

Thus with (46) and (47) we obtain the inequality

$$\|z_T^{(+)}(t) + \Lambda \exp((A+\Lambda)(t-T))\mu^{(\sigma)}\|_X \\ \leq 2\,\exp(-m\,T)\|z_0 - z^{(\sigma)}\|_X + \frac{\exp(-2mT)}{1-\exp(-2m\,T)}\|\Lambda\,\mu^{(\sigma)}\|_X.$$

With (45) for all $t \in [T-t_0,\, T]$ this implies

$$\|\hat{z}_T(t) - z^{(\sigma)} + \exp((A+\Lambda)(t-T))\Lambda\,\mu^{(\sigma)}\|_X \\ \leq 2\,(1+\exp(m\,t_0))\exp(-m\,T)\left[\|z_0 - z^{(\sigma)}\|_X + \|\Lambda\,\mu^{(\sigma)}\|_X\right]$$

and (41) follows. □

Now we state our second main result. We show that also on the first part of the time interval $[0,\, T]$ of the form $[0,\, t_0]$, where $t_0 > 0$ is fixed for large time horizons T the optimal trajectory approaches exponentially fast a limit-trajectory that is independent of the time horizon T.

Theorem 2. *Assume that A and $B\,B^*$ commute and that* (27) *holds. Let Λ be defined as in* (22). *Let $m > 0$ denote a constant such that* (39) *holds. Assume that (40) holds. For $s \in [0, T]$, define the limit trajectory*

$$\Xi(s) = z^{(\sigma)} + \exp((A-\Lambda)s)(z_0 - z^{(\sigma)}).$$

With the notation $\hat{z}_T$ for the optimal state corresponding to the time horizon T for all $t \in [0,\, t_0]$ we have

$$\|\hat{z}_T(t) - \Xi(t)\|_X \leq \frac{3 + 2\exp(m\,t_0)}{\exp(mT)}\left[\|z_0 - z^{(\sigma)}\|_X + \|\Lambda\,\mu^{(\sigma)}\|_X\right]. \tag{49}$$

This means that for $t \in [0,\, t_0]$ the optimal state $\hat{z}_T(t)$ approaches the limit trajectory $\Xi(t)$ exponentially fast with respect to T. Note that the point $\Xi(t)$ only depends on the distance to the initial time zero.

P r o o f. We use the same notation as in the proof of Theorem 1. Note that (40) implies $\|\mathcal{N}(T)\exp(2\,\Lambda\,T)\| \leq 2$. We use the representation (44) of $\hat{z}_T(t) - z^{(\sigma)}$ with $\hat{z}_T^{(+)}(t)$ and $\hat{z}_T^{(-)}(t)$ defined in (34). For all $t \in [0,\, T]$ we have (47) and (48) implies

$$\|\exp((A+\Lambda)t)\,l_+^{(2)}(T)\| \leq 2\,\exp(-m(T-t_0))\|\Lambda\,\mu^{(\sigma)}\|_X.$$

Thus we obtain the inequality

$$\|\hat{z}_T^{(+)}(t)\|_X \leq 2[1+\exp(m\,t_0)]\exp(-mT)\left[\|z_0 - z^{(\sigma)})\|_X + \|\Lambda\,\mu^{(\sigma)}\|_X\right].$$

Using (31) we obtain

$$\begin{aligned}\hat{z}_T^{(-)}(t) - \exp((A-\Lambda)t)(z_0 - z^{(\sigma)}) &= \exp((A-\Lambda)t)\,\left[\frac{1}{2}\,l_-(T) - (z_0 - z^{(\sigma)})\right]\\ &= -\exp((A-\Lambda)t)\,\frac{1}{2}\,l_+(T).\end{aligned}$$

With (42) this yields the inequality

$$\begin{aligned}&\|\hat{z}_T^{(-)}(t) - \exp((A-\Lambda)t)(z_0 - z^{(\sigma)})\|_X\\ &\qquad\leq \frac{\exp(-2m\,T)}{1-\exp(-2m\,T)}\,\left[\|(z_0 - z^{(\sigma)})\|_X + \|\Lambda\mu^{(\sigma)}\|_X\right].\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\|z_T(t) - z^{(\sigma)} - \exp((A-\Lambda)t)(z_0 - z^{(\sigma)})\|_X\\ &\qquad= \|\hat{z}_T^{(+)}(t) + \hat{z}_T^{(-)}(t) - \exp((A-\Lambda)t)(z_0 - z^{(\sigma)})\|_X\\ &\qquad\leq \|\hat{z}_T^{(+)}(t)\| + \|\hat{z}_T^{(-)}(t) - \exp((A-\Lambda)t)(z_0 - z^{(\sigma)})\|_X\\ &\qquad\leq (3+2\exp(m\,t_0))\exp(-mT)\,\left[\|(z_0 - z^{(\sigma)})\|_X + \|\Lambda\mu^{(\sigma)}\|_X\right]\end{aligned}$$

and (49) follows. □

Remark 1. For the optimal adjoint state we have the representation

$$\begin{aligned}\hat{\mu}_T(t) - \mu^{(\sigma)} &= \frac{1}{2}\,\Lambda^{-1}\,(\exp((A+\Lambda)t)l_+(T) - \exp((A-\Lambda)t)l_-(T))\\ &= \Lambda^{-1}\hat{z}_T^{(+)}(t) - \Lambda^{-1}\hat{z}_T^{(-)}(t).\end{aligned}$$

Hence results analogous to Theorem 1 and Theorem 2 hold for the optimal adjoint state $\hat{\mu}_T$. We also find an initial limit trajectory and a terminal limit trajectory for the optimal adjoint states. So we see that the situation for the optimal adjoint state and the optimal control has the same structure. This is similar as for the classical turnpike property, see [4].

Due to (21), we have $\Lambda^{-1}B(\hat{u}_T - u^{(\sigma)}) = \Lambda^{-1}B\,B^*(\hat{\mu}_T - \mu^{(\sigma)}) = \Lambda(\hat{\mu}_T - \mu^{(\sigma)}) = \hat{z}_T^{(+)}(t) - \hat{z}_T^{(-)}(t)$. Hence the proofs of Theorem 1 and Theorem 2 also yields a similar turnpike structure with an initial and terminal limit trajectory for $\Lambda^{-1}B(\hat{u}_T - u^{(\sigma)})$.

6. Example. In this section we discuss the phenomenon of initial and terminal limit trajectories for large time horizons for specific optimal control problems. We consider a finite dimensional example and an example with a distributed parameter system, a system that is governed by the wave equation.

6.1. An example with an ordinary differential equation. Let a matrix $A \in \mathbb{R}^{n\times n}$ be given, $A^T = -A$. Let an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ be given $Q\,Q^T = I$. For a real number γ, let $B = \gamma\,Q$. Then we have

$$BB^T = \gamma^2 I > 0$$

and A and BB^T commute. Hence for the optimal control problem $\mathbf{P}(T)$ defined in (5) that is governed by the ordinary differential equation $z' = Az + Bu$ Theorem 1 and Theorem 2 are applicable and thus (41) and (49) hold.

6.2. An example with the wave equation. Similar as in [11] as an example we consider an optimal control problem for a system that is governed by the wave equation. We can write the second order system as a system of first order differential equations with a skew–adjoint operator A, see [22]. In this example, different from [11] we consider the energy as a part of the objective functional.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2 boundary. Let $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ be given. Let $T > 0$ and $\gamma > 0$ be given. Define $Q = [0,T] \times \Omega$ and $\Sigma = [0,T] \times \partial\Omega$. Let a desired state $y_d \in H^2(\Omega) \cap H_0^1(\Omega)$ be given.

Consider the optimal control problem

$$\begin{cases} \min\limits_{u\in L^2(Q)} \displaystyle\int_Q \left[|y_t|^2 + \|\nabla(y - y_d)\|_{\mathbb{R}^N}^2 + \gamma|u|^2\right] \\ \text{subject to } y(0,\,x) = y_0(x), y_t(0,\,x) = 0 \text{ for } x \in \Omega \text{ a.e.}, \\ y(t,x) = 0 \text{ for } (t,x) \in \Sigma \text{ a.e.}, \\ \text{and } y_{tt} = \Delta y + u \text{ on } Q. \end{cases}$$

For results on the well-posedness of the corresponding initial boundary value problem see [16]. Theorem 1 and Theorem 2 imply that for the optimal state $t \mapsto \hat{y}_T(t)$ for $t > 0$ there exist an initial limit trajectory $t \mapsto \lim\limits_{T\to\infty} \hat{y}_T(t)$ and a terminal limit trajectory $t \mapsto \lim\limits_{T\to\infty} \hat{y}_T(T-t)$.

Assume that $(\varphi_j)_{j=1}^{\infty}$ is a complete orthonormal sequence of eigenfunctions of the Dirichlet–Laplacian $A_0 = -\Delta$, with the homogeneous Dirichlet boundary

conditions on $\partial\Omega$. So for $j \in \{1, 2, 3, \dots\}$ we have $-\Delta\varphi_j = \lambda_j\varphi_j$ with $\lambda_j \geq \lambda_{\min} > 0$. Assume that

$$\frac{1}{4\,\gamma} < \lambda_{\min}. \tag{50}$$

We can find real coefficients β_j and η_j such that $\sum |\beta_j|^2 < \infty$, $\sum |\eta_j|^2 < \infty$, $y_d = \sum_{j=1}^{\infty} \beta_j\,\varphi_j$ and $y_0 = \sum_{j=1}^{\infty} \eta_j\,\varphi_j$. We can represent the state $y(t,\,x)$ in the form $y(t,\,x) = \sum_{j=1}^{\infty} \alpha_j(t)\,\varphi_j(x)$ and the objective functional in terms of the coefficients $\alpha_j(t)$. We obtain

$$\sum_{j=1}^{\infty} \int_0^T |\alpha_j'(t)|^2 + \lambda_j|\alpha_j(t) - \beta_j|^2 + \gamma\,|\alpha_j''(t) + \lambda_j\alpha_j(t)|^2\,dt.$$

So the spectral representation yields a sequence of optimal control problems for $j \in \{1, 2, 3, \dots\}$:

$$\min_{\alpha_j \in H^2([0,T])} \int_0^T |\alpha_j'(t)|^2 + \lambda_j|\alpha_j(t) - \beta_j|^2 + \gamma\,|\alpha_j''(t) + \lambda_j\alpha_j(t)|^2\,dt$$

subject to $\alpha_j(0) = \eta_j,\ \alpha_j'(0) = 0$.

For $t \in [0, T]$, the optimality conditions yield the differential equation

$$\alpha_j^{(4)}(t) + \left(2\lambda_j - \frac{1}{\gamma}\right)\,\alpha_j^{(2)}(t) + \lambda_j\left(\lambda_j + \frac{1}{\gamma}\right)\alpha_j(t) = \lambda_j\frac{1}{\gamma}\,\beta_j$$

with the boundary conditions

$$\alpha_j(0) = \eta_j,\ \alpha_j'(0) = 0,\ \alpha_j^{(3)}(T) = \left(\frac{1}{\gamma} - \lambda_j\right)\alpha_j'(T),\ \alpha_j''(T) = -\lambda_j\alpha_j(T).$$

Define the characteristic polynomial $p_j(z) = z^4 + \left(2\lambda_j - \frac{1}{\gamma}\right)\,z^2 + \lambda_j\left(\lambda_j + \frac{1}{\gamma}\right)$. Due to (50) we have $p_j(z) = (z-\omega_j)(z+\omega_j)(z-\bar{\omega}_j)(z+\bar{\omega}_j)$ where $\pm\omega_j$ and $\pm\bar{\omega}_j$ denote the roots of p_j. We have $|\omega_j|^4 = \lambda_j(\lambda_j + \frac{1}{\gamma})$, $Re(\omega_j^2) = -(\lambda_j - \frac{1}{2\gamma})$ and $|Im(\omega_j^2)| = \frac{1}{\gamma}\sqrt{2\gamma\,\lambda_j - \frac{1}{4}}$. Moreover we have

$$|\mathrm{Re}(\omega_j)| \geq \frac{1}{2\sqrt{\gamma}}. \tag{51}$$

For the sake of simplicity we consider the case $y_d = 0$, that is $\beta_j = 0$.

We obtain the representation for $t \mapsto \alpha_j(T;t)$ with fixed T:

$$\alpha_j(T;t) = A_j(T)[(\bar{\omega}_j^2 + \lambda_j)\cosh(\omega_j(T-t)) - (\omega_j^2 + \lambda_j)\cosh(\bar{\omega}_j(T-t))]$$

$$+B_j(T)\left[\bar{\omega}_j\left(\bar{\omega}_j^2 + \lambda_j - \frac{1}{\gamma}\right)\sinh(\omega_j(T-t)) - \omega_j\left(\omega_j^2 + \lambda_j - \frac{1}{\gamma}\right)\sinh(\bar{\omega}_j(T-t))\right]$$

where the coefficients $T \mapsto A_j(T)$, $T \mapsto B_j(T)$ have to be chosen such that $\alpha_j(T;0) = \eta_j$ and $\alpha_j'(T;0) = 0$. Note that the corresponding Wronski-matrix is regular. Define the denominator

$$N_j(T) = |\omega_j|^2\left[\frac{4\lambda_j}{\gamma} + \left(\frac{4\lambda_j}{\gamma} - \frac{1}{\gamma^2}\right)\cosh(\omega_j T)\cosh(\bar{\omega}_j T)\right]$$

$$+\lambda_j\left(\frac{4\lambda_j}{\gamma} + \frac{1}{\gamma^2}\right)\sinh(\omega_j T)\sinh(\bar{\omega}_j T).$$

Condition (50) implies $N_j(T) > 0$. We have

$$A_j(T) = -\eta_j\frac{|\omega_j|^2\left[\left(\bar{\omega}_j^2 + \lambda_j - \frac{1}{\gamma}\right)\cosh(\omega_j T) - \left(\omega_j^2 + \lambda_j - \frac{1}{\gamma}\right)\cosh(\bar{\omega}_j T)\right]}{N_j(T)},$$

$$B_j(T) = -\eta_j\frac{-\omega_j(\bar{\omega}_j^2 + \lambda_j)\sinh(\omega_j T) + \bar{\omega}_j(\omega_j^2 + \lambda_j)\sinh(\bar{\omega}_j T)}{N_j(T)}.$$

The representations of $A_j(T)$ and $B_j(T)$ allow for the asymptotic analysis for $T \to \infty$. Due to (51) we have

$$\lim_{T\to\infty} A_j(T) = 0, \qquad \lim_{T\to\infty} B_j(T) = 0.$$

For $t > 0$ this yields

$$\lim_{T\to\infty} \alpha_j(T; T-t) = 0.$$

For $y_d = 0$ the optimal static state is zero and it is attained with the minimal control cost zero. Thus the optimal control stabilizes the state towards zero until the end of the time interval $[0, T]$. Note that the values $|\alpha_j^{(T)}(T-t)|$ decay exponentially fast with growing T with the order $\exp(-|Re(\omega_j)|\,T)$ and hence at least with the order $\exp\left(-\frac{1}{2\sqrt{\gamma}}\,T\right)$.

7. Conclusions. We have shown that for a certain class of optimal control problems for large time horizons the optimal states approach limit trajectories both in the first part of the time interval and in the last part of the time interval. The limit trajectories are independent of the time horizon T and are approached exponentially fast with increasing time horizon T.

We have shown the results for a class of linear quadratic optimal control problems with a skew-adjoint generator. We expect that our results can be generalized in several directions. For example, a generalization to the case of convex objective functionals similar as in [8] should be possible. In [8] also additional control constraints and state constraints are considered that are independent of the time horizons. Constraints of this type should also be integrated in the turnpike analysis of initial and terminal limit trajectories. The case where the control only acts locally on the system is not covered by our current setting. We expect that under suitable assumptions it is possible to show an analogous result also in this framework.

The generalization of the results to the case of semilinear systems is also a topic of future research. If the nonlinear source term is of a dissipative nature, the proofs should have a similar structure but the linearization of the nonlinear source term appears in the adjoint equation in the necessary optimality conditions. Moreover, additional smallness assumptions for the initial state in appropriate function spaces are in general require to guarantee that the system is well-posed. Assumptions of this type are also required in the case of quasilinear systems. However, in order to guarantee that the solutions remain regular also for large time horizons, additional constraints are necessary. In [10] state constraints with respect to the C^1-norm are included in the optimal control problem. The constraints assure that the system state is a classical solution of the partial differential equation and no shocks can occur.

Also a generalization to games is of interest. We refer to [24] for the study of the classical turnpike phenomenon for discrete time games and to [13] for the definition of a networked boundary control game with the wave equation. The optimal control of the wave equation with measure valued controls has been studied in [15]. It is an interesting question for future research whether also for this problem a similar turnpike structure occurs for large time horizons.

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REFERENCES

[1] S. J. Bernau. The square root of a positive self-adjoint operator. *J. Austral. Math. Soc.* **8** (1968), 17–36.

[2] T. Breiten, L. Pfeiffer. On the turnpike property and the receding-horizon method for linear-quadratic optimal control problems. *SIAM J. Control Optim.* **58**, *2* (2020), 1077–1102, doi: 10.1137/18M1225811.

[3] T. Faulwasser, K. Flasskamp, S. Ober-Blöbaum, M. Schaller, K. Worthmann. Manifold turnpikes, trims, and symmetries. *Math. Control Signals Systems* **34**, *4* (2022), 759–788, doi: 10.1007/s00498-022-00321-6.

[4] T. Faulwasser, L. Grüne, J.-P. Humaloja, M. Schaller. Inferring the adjoint turnpike property from the primal turnpike property, 2021 60th IEEE Conference on Decision and Control (CDC), Austin, TX, USA, 2021, 2578–2583, doi: 10.1109/CDC45484.2021.9683079.

[5] T. Faulwasser, L. Grüne. Turnpike properties in optimal control: An overview of discrete-time and continuous-time results. In: Handbook of Numerical Analysis, vol. **23** (eds E. Trélat, E. Zuazua), 2022, 367–400.

[6] B. Geshkovski, E. Zuazua. Turnpike in optimal control of pdes, resnets, and beyond. Acta Numer. **31** (2022), 135–263, doi: 10.1017/S0962492922000046.

[7] L. Grüne, M. Schaller, A. Schiela. Abstract nonlinear sensitivity and turnpike analysis and an application to semilinear parabolic PDEs. *ESAIM Control Optim. Calc. Var.* **27**, *56* (2021), Paper No. 56, 28 pp, doi: 10.1051/cocv/2021030.

[8] M. Gugat. A turnpike result for convex hyperbolic optimal boundary control problems. *Pure Appl. Funct. Anal.* **4**, *4* (2019), 849–866.

[9] M. Gugat, F. M. Hante. On the turnpike phenomenon for optimal boundary control problems with hyperbolic systems. *SIAM J. Control Optim.* **57**, *1* (2019), 264–289, doi: 10.1137/17M1134470.

[10] M. Gugat, M. Herty. A turnpike result for optimal boundary control of gas pipeline flow. 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2022): 12–16 September 2022, Bayreuth, Germany, `https://epub.uni-bayreuth.de/id/eprint/6809/1/MTNS2022_ExtendedAbstracts_2022-12-22.pdf`.

[11] M. Gugat, A. Keimer, G. Leugering. Optimal distributed control of the wave equation subject to state constraints. *ZAMM, Z. Angew. Math. Mech.* **89** *6* (2009), 420–444, doi: 10.1002/zamm.200800196.

[12] M. Gugat, M. Schuster, E. Zuazua. The finite-time turnpike phenomenon for optimal control problems: stabilization by non-smooth tracking terms. In: Stabilization of distributed parameter systems: design methods and applications, ICIAM 2019 SEMA SIMAI Springer Ser., vol. **2**. Cham, Springer, 2021, 17–41, doi: 10.1007/978-3-030-61742-4_2.

[13] M. Gugat, S. Steffensen. Dynamic boundary control games with networks of strings. *ESAIM Control Optim. Calc. Var.* **24**, *4* (2018), 1789–1813, doi: 10.1051/cocv/2017082.

[14] M. Herty, A. N. Sandjo. On optimal treatment planning in radiotherapy governed by transport equations. *Math. Models Methods Appl. Sci.* **21**, *2* (2011), 345–359, doi: 10.1142/S0218202511005076.

[15] K. Kunisch, P. Trautmann, B. Vexler. Optimal control of the undamped linear wave equation with measure valued controls. *SIAM J. Control Optim.* **54**, *3* (2016), 1212–1244, doi: 10.1137/141001366.

[16] I. Lasiecka, J. L. Lions, R. Triggiani. Nonhomogeneous boundary value problems for second order hyperbolic operators. *J. Math. Pures Appl. (9)* **65**, *2* (1986), 149–192.

[17] I. Lasiecka, R. Triggiani. Control theory for partial differential equations: continuous and approximation theories. II. Abstract hyperbolic-like systems over a finite time horizon. Encyclopedia Math. Appl., vol. **75**. Cambridge, Cambridge University Press, 2000, doi: 10.1017/CBO9780511574801.002.

[18] J.-L. Lions. Optimal control of systems governed by partial differential equations. Die Grundlehren der mathematischen Wissenschaften, Band **170**. Translated from the French by S. K. Mitter. New York-Berlin, Springer-Verlag, 1971.

[19] P. A. Samuelson. A catenary turnpike theorem involving consumption and the golden rule. *The American Economic Review* **55**, *3* (1965), 486–496.

[20] Z. Sebestyén, Z. Tarcsay. On the square root of a positive selfadjoint operator. *Period. Math. Hungar.* **75**, *2* (2017), 268–272, doi: 10.1007/s10998-017-0192-1.

[21] E. Trélat, C. Zhang. Integral and measure-turnpike properties for infinite-dimensional optimal control systems. *Math. Control Signals Systems* **30**, *1* (2018), Art, 3, 34 pp, doi: 10.1007/s00498-018-0209-1.

[22] M. Tucsnak, G. Weiss. Observation and control for operator semigroups. Birkhäuser Adv. Texts, Basler Lehrbüch [Birkhäuser Advanced Texts: Basel Textbooks]. Basel: Birkhäuser Verlag, 2009.

[23] A. Wouk. A note on square roots of positive operators. *SIAM Rev.* **8** (1966), 100–102, doi: 10.1137/1008008.

[24] A. J. Zaslavski. Discrete-Time Optimal Control and Games on Large Intervals. Springer Optim. Appl., vol. **119**. Cham, Springer, 2017, doi: 10.1007/978-3-319-52932-5.

[25] A. J. Zaslavski. Turnpike conditions in infinite dimensional optimal control. Springer Optim. Appl., vol. **148**. Cham, Springer, 2019, doi: 10.1007/978-3-030-20178-4.

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